THE UNIVERSITY OF ALBERTA

PSEUDO-PLASTIC FLOW

by

© DAVID B. CRAIG

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MECHANICAL ENGINEERING

EDMONTON, ALBERTA

Fall, 1970

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "PSEUDO-PLASTIC FLOW" submitted by David B. Craig in partial fulfilment of the requirements for the degree of Master of Science.

Date ....May 26, 1970.....

ABSTRACT

The topic of this thesis is the flow of pseudo-plastic materials. A discussion of the empirical model used to describe the behavior of this type of material is presented, and objections to its use are discussed. It is shown that the incompressible Newtonian fluid and the rigid-perfectly plastic von Mises solid are special cases of this model. Two problems, plane flow in a converging channel and axi-symmetric flow in a cone, are solved for various "degrees" of non-Newtonian behavior. Body forces and thermal effects are neglected, and the flow is assumed to be fully developed. A numerical technique developed by Clutter and Smith to solve boundary-layer problems is adapted to solve the non-linear ordinary differential equations that govern the flow problems considered here.

To obtain the solution for fully developed flow, the streamlines are assumed to be radial, meeting at the virtual apex of the converging channel or cone. An attempt was made, with no success, to consider entrance effects for the flow of a Newtonian fluid in a cone; this, and the problem of entrance effects for the pseudo-plastic material, is discussed.

ACKNOWLEDGEMENTS

The author wishes to thank his supervisor, Dr. J.B. Haddow, for his excellent guidance of this research. Thanks are also due to Miss Susan Schultz for typing this thesis.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Figure | Page

NOTATION

Where cartesian tensor notation is used the usual summation convention is employed

σ_{ij}	cartesian components of the stress tensor referred to axes OX_i
$\sigma_{ij,k}$	comma denotes partial differentiation with respect to x_k
v_i	components of the velocity vector
d_{ij}	components of the rate of strain tensor
s_{ij}	components of the deviatoric stress tensor
μ	a rheological parameter in the power law model and the viscosity of a Newtonian fluid
I_1	d_{ii}, a strain rate invariant
I_2	$\frac{1}{2} d_{ij}d_{ij}$, a strain rate invariant
I_3	$\frac{1}{3} d_{im}d_{mn}d_{ni}$, a strain rate invariant
k	shear yield stress for von Mises perfectly plastic solid
α	semi-angle of the channel or cone
$-p$	the hydrostatic part of the stress tensor
Q	volume flow rate
P	the hydrostatic stress
α_1, α_2	rheological parameters of the Stokesian fluid, called the apparent viscosity and cross viscosity, or viscous normal stress coefficient, respectively

E work function

ψ Stokes' stream function

CHAPTER I

INTRODUCTION

Rheology is a branch of continuum mechanics concerned with the study of anything that flows. "A continuum is a (hypothetical) structureless substance to which, at any point, we can assign kinematical or dynamical variables which are continuous functions of the spatial coordinates of that point" (1). The solution of a flow problem must, therefore, begin with a consideration of the basic equations of continuum mechanics.

Continuum mechanics is based on the momentum equations, the equations of continuity and energy, and constitutive equations. Assuming there are no body moments, the stress tensor is symmetric with six independent components. The energy equation is not considered since thermal effects are neglected in this thesis. Neglecting body forces and considering incompressible quasi-static flow, the equations of motion and continuity are, respectively,

$$\sigma_{ij,j} = 0 , \tag{1.1}$$

and

$$v_{i,i} = 0 . \tag{1.2}$$

Equations (1.1) and (1.2) provide four equations for the nine unknowns σ_{ij} and v_i. This is all that can be determined

from kinematics and dynamics. Constitutive equations describe the intrinsic behavior of a material in terms of stress, stress rates, deformation, and deformation rates, and provide the additional equations needed to solve problems. The requirements for constitutive equations are discussed by Aris (2). The constitutive equations used in this thesis are discussed in the next chapter. Equations (1.1), (1.2), and the constitutive equations, along with the appropriate boundary conditions, are sufficient to solve the problems considered in this thesis.

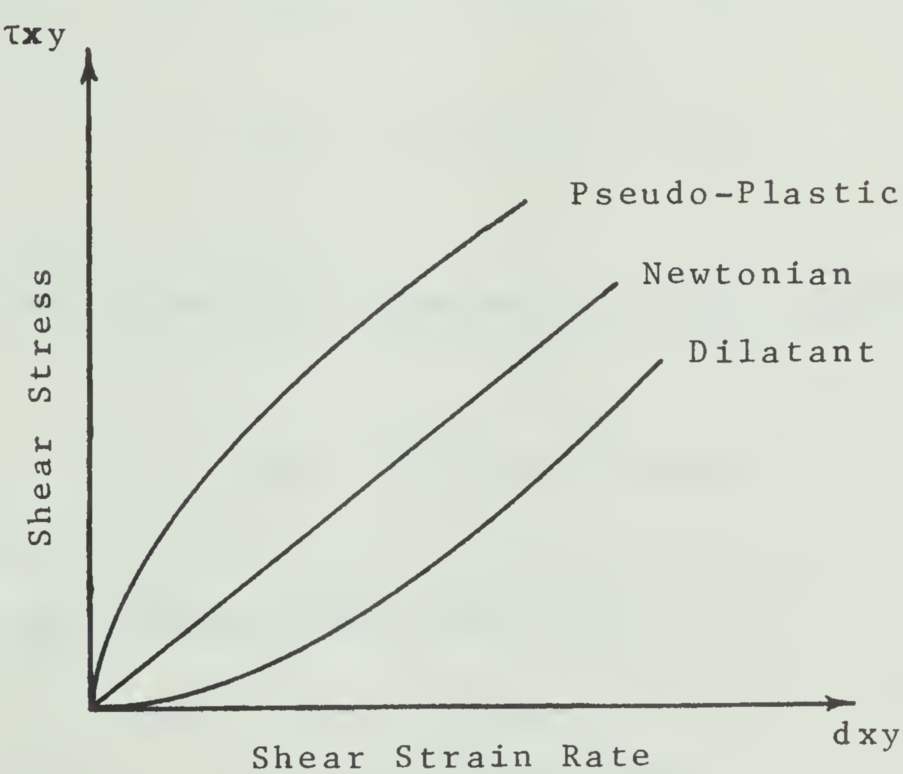

FIGURE 1.1

STRESS-STRAIN RELATIONSHIPS
IN SIMPLE SHEAR

1.1 Pseudo-Plasticity

Fluids which have an apparent viscosity which increases with increasing strain rate, as measured in simple shear flow, are called dilatant, and those with an apparent viscosity which decreases with increasing strain rate are called pseudo-plastic. The name dilatant is actually a misnomer, since it implies an increase in volume. Brodkey (3) reserves this terminology for materials which actually dilate, and calls the type of behavior considered here shear-thickening. A qualitative picture of the behavior of these two types of fluids is given in Figure 1.1.

The model used in this thesis to describe pseudo-plastic flow is the power law model,

$$S_{ij} = 2\mu \, (2I_2)^{\frac{n-1}{2}} \, d_{ij} \;, \tag{1.3}$$

where the d_{ij} are the components of the strain rate tensor,

$$d_{ij} = \frac{1}{2} \, (v_{i,j} + v_{j,i}) \;,$$

and I_2 is a strain rate invariant,

$$I_2 = \frac{1}{2} \, d_{ij} \, d_{ij} \;.$$

This model is a generalization of the model suggested by Ostwald for simple shear flow, and represents dilatant fluids for $n > 1$, and pseudo-plastic fluids for $n < 1$. For $n = 1$

it represents the incompressible Newtonian fluid, and for $n = 0$ the rigid-perfectly plastic von Mises solid, with constitutive equation

$$d_{ij} = \lambda S_{ij} , \tag{1.4}$$

where

$$\lambda = \sqrt{\frac{d_{mn}\, d_{mn}}{k}} ,$$

and

$$k = \sqrt{2}\ \mu$$

Only pseudo-plastic materials, and the special cases of the Newtonian fluid and the von Mises solid, are considered in this thesis.

1.2 Known Solutions

The problems considered are plane flow in a converging channel and axi-symmetric flow in a cone. Solutions to these two problems have been obtained for the special cases of the Newtonian fluid and the rigid-perfectly plastic von Mises solid.

For the Newtonian fluid, the effect of the inclusion of inertia terms has been considered for both problems. Rosenhead (4) obtained solutions in terms of elliptic functions for the plane flow problem with inertia effects included. These solutions indicate that, for a given semi-angle and

Reynolds number, an infinite number of velocity profiles are possible. These profiles may or may not be symmetric with respect to the center line of the channel, and inflow and outflow can occur at the same time. Rosenhead suggested that stability considerations would probably exclude many of these flows, and that the flow pattern obtained in an experiment would be determined by the pressure conditions at the inlet and outlet. Hamel (5) showed that no radial flow solution exists for the axi-symmetric problem if inertia terms are included. Simple radial flow solutions are obtained for both problems when inertia effects are neglected. These solutions are outlined below.

The solution for the plane, quasi-static flow of a Newtonian fluid in a converging channel with semi-angle α is easily found to be

$$v_r = \frac{2Q \sin^2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha} \left[1 - \left(\frac{\sin\theta}{\sin\alpha}\right)^2\right] \frac{1}{r} , \qquad (1.5)$$

and

$$p - p_0 = \frac{4\mu Q}{\sin 2\alpha - 2\alpha\cos 2\alpha} \left[\frac{(\sin^2\theta - 3)}{r^2} - \frac{(\sin^2\theta_0 - 3)}{r_0{}^2}\right] ,$$

where p_0 is the hydrostatic pressure at θ_0, r_0, Q is the volume flow rate,

$$Q = 2r \int_0^\alpha v_r \, d\theta ,$$

and μ is the coefficient of viscosity.

The corresponding solution for the axi-symmetric problem was given by Bond (6). This solution can be written in the form

$$v_r = \frac{3Q \sin^2\alpha}{2\pi(1-\cos\alpha)^2 \ (1+2\cos\alpha)} \left[1 - \left(\frac{\sin\theta}{\sin\alpha}\right)^2\right] \frac{1}{r^2} , \qquad (1.6)$$

and

$$p - p_0 = \frac{\mu Q}{\pi(1-\cos\alpha)^2 \ (1+2\cos\alpha)} \left[\frac{(2-3\sin^2\theta)}{r^3} - \frac{(2-3\sin^2\theta_0)}{r_0{}^3}\right] ,$$

where

$$Q = 2\pi r^2 \int_0^\alpha v_r \sin\theta \, d\theta .$$

Nadai obtained the stress field for the flow of a rigid-perfectly plastic von Mises solid in a converging channel, with the assumption that the shear yield stress k is generated at the sides. The associated velocity field was found by Hill (7). A shear stress boundary condition at the channel wall was used in solving this problem, in contrast to the no-slip boundary condition used in obtaining the solutions for the Newtonian fluid.

Shield (8) considered the flow of a rigid-perfectly plastic solid in a cone. Shield did not obtain a closed form solution, but integrated the equations numerically.

Again, a shear stress boundary condition was used.

These known solutions, which are special cases of the flows studied in this thesis, provide a check on the results obtained, and the solutions for the Newtonian fluid are used as a starting point in the numerical procedure.

1.3 Aim of This Thesis

The aim of this thesis is to study pseudo-plastic flow in a converging channel and in a cone, and to develop an efficient numerical technique for solving the non-linear two-point boundary-value problems for fully developed flow.

CHAPTER II

THE POWER LAW MODEL

In this chapter, the power law is considered as an empirical approximation for the constitutive equations of certain Stokesian fluids. This model, along with the equations of motion and continuity, is used to describe pseudoplastic flow.

2.1 The Stokesian Fluid

The Stokesian Fluid is an isotropic, homogeneous, inelastic fluid which satisfies the following relations (9), (10);

$$\sigma_{ij} = \sigma_{ij}(d_{rs}) \ ,$$

and

$$\sigma_{ij}(0) = - P \ \delta_{ij} \ , \tag{2.1}$$

where P is the hydrostatic stress. The components of the stress tensor can be written as the sum of a deviatoric part and a hydrostatic part,

$$\sigma_{ij} = S_{ij} - P \ \delta_{ij} \ , \tag{2.2}$$

where the S_{ij} are components of the deviatoric stress tensor, and

$$P = -\frac{\sigma_{kk}}{3} .$$

Assuming the components of the viscous stress tensor can be expressed as polynomials in the components of the rate of strain tensor, the constitutive equations for the incompressible Stokesian fluid may be written as

$$\sigma_{ij} = -P\,\delta_{ij} + \alpha_1\, d_{ij} + \alpha_2\, d_{im}\, d_{mj} , \tag{2.3}$$

since, by using the Cayley-Hamilton theorem, all terms of higher order can be written in terms of δ_{ij}, d_{ij}, $d_{im}\, d_{mj}$, and the two non-vanishing invariants of the rate of strain tensor,

$$I_2 = \frac{1}{2}\, d_{ij}\, d_{ij} ,$$

and

$$I_3 = \frac{1}{3}\, d_{im}\, d_{mn}\, d_{ni} .$$

The parameters α_1 and α_2 are functions of I_2 and I_3, and must be such that the rate of viscous dissipation is non-negative,

$$T_{ij}\, d_{ij} = \alpha_1\, d_{ij}\, d_{ij} + \alpha_2\, d_{im}\, d_{mj}\, d_{ij}$$

$$= 2\alpha_1\, I_2 + 3\alpha_2\, I_3 \geq 0 .$$

The constitutive equations of the Stokesian fluid are non-linear, and its rheological parameters depend on the strain rate. Also, these fluids exhibit normal stress effects if $\alpha_2 \neq 0$. Equations (2.3) are the general form of the constitutive equations for the incompressible Stokesian fluid.

2.2 The Power Law

The mathematical difficulties encountered in applying equation (2.3) are extreme. Also, experimentalists have not, as yet, been able to determine the dependence of α_1 on I_3, or to measure α_2 unambiguously. Consequently simpler relations have been suggested. Some of these are given by Bird (11).

For some simple flow problems normal stress effects have no influence on the relationship between shear stress and shear rate. If the relationship between shear stress and shear rate alone is of interest, it is reasonable to assume that α_2 is zero. If the existence of a work function

$$E = E(d_{ij})$$

is assumed (12) such that

$$S_{ij} = \frac{\partial E(d_{rs})}{\partial d_{ij}} ,$$

then, for an incompressible fluid,

$$E = E(I_2, I_3) \ ,$$

and

$$S_{ij} = \frac{\partial E}{\partial d_{ij}} = \frac{\partial E}{\partial I_2} d_{ij} + \frac{\partial E}{\partial I_3} (d_{ik} d_{kj} - \frac{2}{3} I_2 \delta_{ij}) \ .$$

For an incompressible Stokesian fluid

$$S_{ij} = \alpha_1 d_{ij} + \alpha_2 (d_{ik} d_{kj} - \frac{2}{3} I_2 \delta_{ij})$$

where

$$\frac{\partial E}{\partial I_2} = \alpha_1 (I_2, I_3) \ ,$$

and

$$\frac{\partial E}{\partial I_3} = \alpha_2 (I_2, I_3) \ .$$

The assumption that α_2 is zero therefore implies that

$$E = E(I_2) \ ,$$

$$\alpha_1 = \alpha_1 (I_2) \ ,$$

and

$$P = p = - \frac{\sigma_{kk}}{3} \ .$$

Hence, the assumption that the coefficient α_1 in the power law,

$$\alpha_1 = 2\mu(2I_2)^{\frac{n-1}{2}} ,$$

does not depend on the third strain rate invariant has some theoretical justification.

Apparent viscosity can be measured using a Couette or capillary viscometer. Since I_3 is zero for the flow in these instruments, the dependence of α_2 on I_3 can not be determined from these tests. Tanner (13) proposed the use of quasi-static flow in a converging cone, in which I_3 is non-zero, to check the validity of the power law. If the work function exists, the non-zero value of I_3 will not affect the flow, and the velocity profile and pressure will agree with that predicted by the power law if the constants n and 2μ have been chosen to fit the power law to the results obtained from viscometer tests. Approximate values for n and 2μ for several fluids are given in (14).

Reiner (15) raised three objections to the power law model. First, for zero strain rates and n less than one, the apparent coefficient of viscosity is infinite, and second, for infinite strain rates, it is zero. Since neither of these extremes is of interest in practical analysis, these objections are not serious. Fredrickson (16) shows that the power law can be made to fit experimental data over a certain range of rates of deformation. Thus, despite the above objections, the power law is valid if the rates of deformation are suitably restricted. The third objection is more serious. The dimensions of the parameter μ depend

on n, and consequently the power law can not be a physical law. However, if n is a constant for a given fluid, and μ is determined from experimental data, then the power law is a valid empirical relationship.

CHAPTER III

PSEUDO-PLASTIC FLOW

In this chapter the governing equations for the flow problems under consideration are derived.

3.1 Plane Flow in a Converging Channel

The first problem considered is plane, quasi-static flow of an incompressible material in a converging channel. Plane polar coordinates, r and θ (Figure 3.1), are used, and the flow is assumed to be radial. The assumption of radial flow is justified since a non-trivial solution is obtained.

For radial flow, the components of the rate of strain tensor are, in polar coordinates,

$$d_r = \frac{\partial v_r}{\partial r} ,$$

$$d_\theta = \frac{v_r}{r} , \tag{3.1}$$

and

$$d_{r\theta} = \frac{1}{2r}\frac{\partial v_r}{\partial \theta} .$$

The continuity equation is

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} = 0 . \tag{3.2}$$

Equation (3.2) requires that

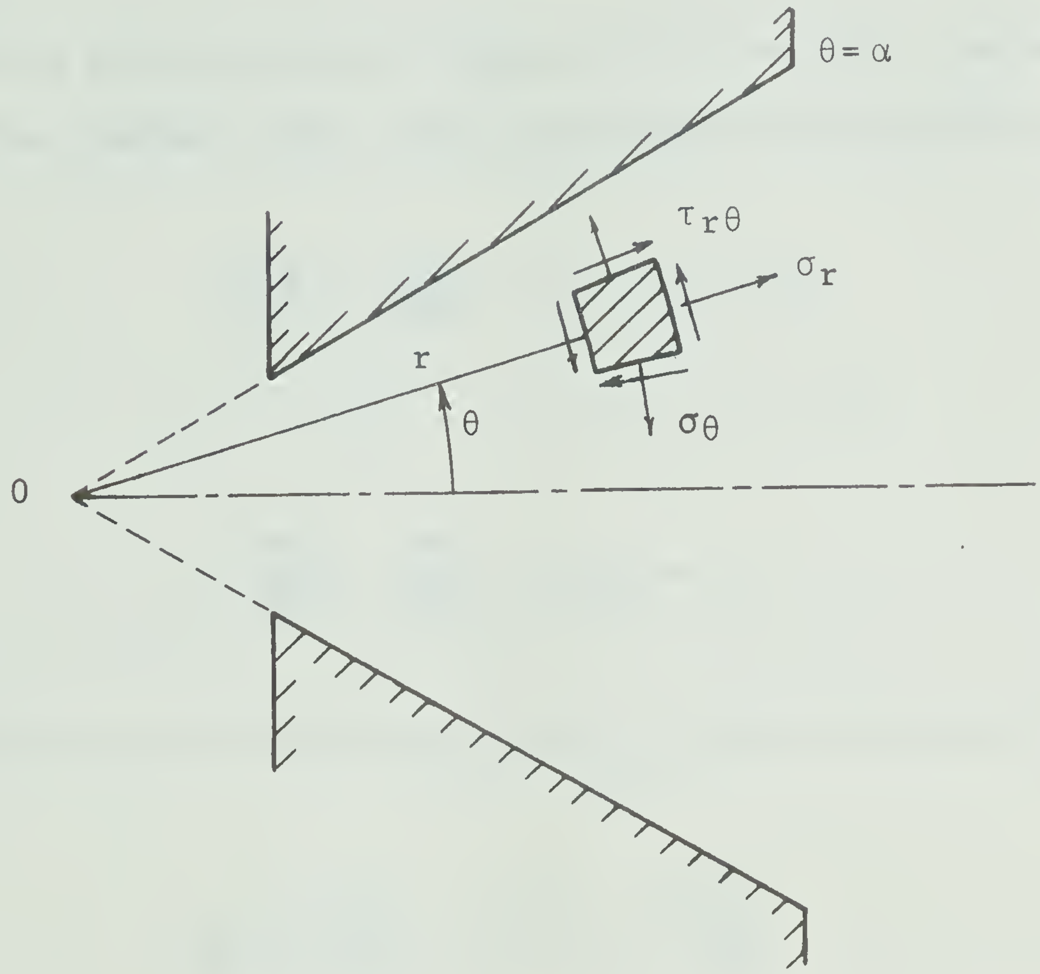

FIGURE 3.1

FLOW IN A CONVERGING CHANNEL

$$v_r = \frac{g(\theta)}{r} \quad . \tag{3.3}$$

Substitution of equation (3.3) into equations (3.1) gives

$$d_r = - d_\theta = - \frac{g(\theta)}{r^2} \ ,$$

and

$$d_{r\theta} = \frac{g'(\theta)}{2r^2} , \tag{3.4}$$

where the prime denotes differentiation with respect to θ.

For radial flow, the equations of motion become

$$r \frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{r\theta}}{\partial \theta} + \sigma_r - \sigma_\theta = 0 ,$$

and

$$r \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_\theta}{\partial \theta} + 2\sigma_{r\theta} = 0 .$$

Substitution of equations (2.2) in these equations yields

$$\frac{\partial \rho}{\partial r} = \frac{\partial S_r}{\partial r} + 2 \frac{S_r}{r} + \frac{1}{r} \frac{\partial S_{r\theta}}{\partial \theta} ,$$

and

$$\frac{\partial \rho}{\partial \theta} = - \frac{\partial S_r}{\partial \theta} + 2S_{r\theta} + r \frac{\partial S_{r\theta}}{\partial r} , \tag{3.5}$$

since

$$S_\theta = - S_r .$$

The hydrostatic pressure can be eliminated from equations (3.5). This gives the single partial differential equation

$$2 \frac{\partial^2 S_r}{\partial r \partial \theta} + \frac{2}{r} \frac{\partial S_r}{\partial \theta} + \frac{1}{r} \frac{\partial^2 S_{r\theta}}{\partial \theta^2} - r \frac{\partial^2 S_{r\theta}}{\partial r^2} - 3 \frac{\partial S_{r\theta}}{\partial r} = 0 . \tag{3.6}$$

Substitution of equations (3.4) into the power law yields

$$S_r = - S_\theta = - \frac{2^{\frac{3-n}{2}}}{r^{2n}} \mu g [4 g^2 + (g')^2]^{\frac{n-1}{2}} ,$$

and

$$S_{r\theta} = \frac{2^{\frac{1-n}{2}}}{r^{2n}} \mu g [4 g^2 + (g')^2]^{\frac{n-1}{2}} ,$$

for the deviatoric components of the stress tensor. Let

$$m(\theta) = \frac{g(\theta)}{V} , \tag{3.7}$$

where m is a non-dimensional velocity depending on θ only, and

$$V = g(0)$$

is a constant to be determined from the volume flow rate.

With the substitution (3.7) the stress deviators become

$$S_r = - S_\theta = - \frac{2^{\frac{3-n}{2}}}{r^{2n}} \mu V^n mG ,$$

and

$$S_{r\theta} = \frac{2^{\frac{1-n}{2}}}{r^{2n}} \mu V^n m'G , \tag{3.8}$$

where

$$G = [4m^2 + (m')^2]^{\frac{n-1}{2}} .$$

Substitution of equations (3.8) in equation (3.6) yields

$$(m'G)'' + 4n(1-n)\ m'G + 4(2n-1)(mG)' = 0 . \tag{3.9}$$

Substituting equations (3.8) in equations (3.5) and integrating gives the hydrostatic pressure

$$p = \frac{2^{\frac{1-n}{2}}}{r^{2n}} \mu V^n \left[\frac{2}{n}(1-n)\ mG - \frac{1}{2n}(m'G)'\right] + A , \tag{3.10}$$

where A is an arbitrary constant.

For n = 1 equations (3.9) and (3.10) reduce to the equations for quasi-static flow of a Newtonian fluid in a converging channel. De Vries (17) showed that equation (3.9), with n = 0, reduces to the equation found by Nadai for the flow of a rigid-perfectly plastic von Mises solid in a converging channel. Equation (3.10) can be rearranged to eliminate n from the denominator. Dividing equation (3.9) by 2n, integrating, and rearranging, yields

$$\frac{2}{n}(1-n)\ mG - \frac{1}{2n}(m'G)' = 2mG + 2(1-n)\int_0^\theta m'G\ d\theta + D ,$$

where D is a constant. The expression for the hydrostatic pressure becomes, for $n = 0$,

$$p = \sqrt{2}\,\mu[2mG + 2\int_0^\theta m'G\,d\theta] + E(r) + A .$$

This is equivalent to the expression found for the pressure from Nadai's equations, for with the substitution

$$\frac{m'}{2m} = -\tan 2\psi$$

De Vries showed that equation (3.9) reduces to

$$\psi' = c \sec 2\psi - 1 .$$

Using this substitution the expression for the pressure becomes

$$p = \sqrt{2}\,\mu[-\cos 2\psi + 2\int_0^\psi \frac{\cos 2\psi \sin 2\psi\,d\psi}{c-\cos 2\psi}] + E(r) + A,$$

or, by performing the integration,

$$p = \sqrt{2}\,\mu\, c \ln(c-\cos 2\psi) = E(r) + A . \qquad (3.11)$$

The function E(r) can be evaluated since, from equation (3.10),

$$\left(\frac{\partial p}{\partial r}\right)_{n=0} = \frac{\sqrt{2}\,\mu}{r}\,[(m'G)' - 4mG],$$

which, with the above substitution, becomes

$$\left(\frac{\partial p}{\partial r}\right)_{n=0} = 2\,\frac{\sqrt{2}\mu}{r}\,c\ .$$

From equation (3.11),

$$\left(\frac{\partial p}{\partial r}\right)_{n=0} = E'(r)\ ,$$

and therefore

$$E(r) = 2\sqrt{2}\,\mu c \ln r + B.$$

The pressure now becomes

$$p = \sqrt{2}\,\mu[c \ln(c-\cos 2\psi) + 2c \ln r] + F\ ,$$

which is the expression derived from Nadai's equations. These results show that these two problems are special cases of the flows considered in this thesis.

The problem reduces to the solution of equation (3.9) subject to the boundary conditions

$$m(0) = 1\ ,$$

$$m'(0) = 0$$

from symmetry, and the no slip boundary condition

$$m(\alpha) = 0 ,$$

where α is the semi-angle of the channel. In general, this must be done numerically.

For $n = \frac{1}{2}$, De Vries (17) showed that equation (3.9) can be integrated analytically for $m'G$. The solution obtained is

$$m'G = c \sin\theta ,$$

and the shear stress becomes

$$S_{r\theta} = \frac{2^{\frac{1}{4}} \mu V^{\frac{1}{2}}}{r} c \sin\theta .$$

This solution is used to check the accuracy of the numerical solution.

3.2 Axi-Symmetric Flow in a Cone

The second problem considered is the axi-symmetric, quasi-static flow of an incompressible material in a cone. Again, the flow is assumed to be radial. Spherical coordinates r, ϕ, and θ (Figure 3.2), are used.

The non zero components of the rate of strain tensor,

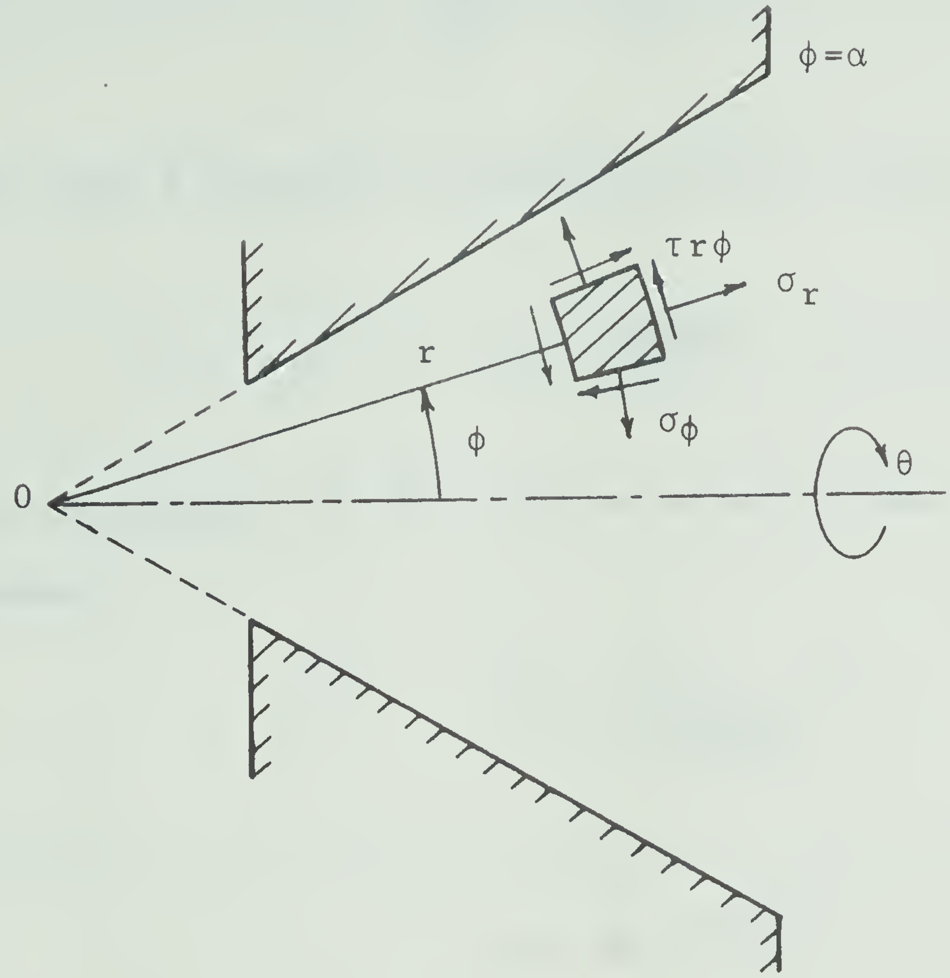

FIGURE 3.2

FLOW IN A CONE

in spherical coordinates, are

$$d_r = \frac{\partial v_r}{\partial r} ,$$

$$d_\phi = d_\theta = \frac{v_r}{r} , \qquad (3.12)$$

and

$$d_{r\phi} = \frac{1}{2r} \frac{\partial v_r}{\partial \phi}.$$

From the continuity equation,

$$\frac{\partial v_r}{\partial r} + 2\,\frac{v_r}{r} = 0 \; ,$$

the radial velocity is found to be of the form

$$v_r = \frac{g(\phi)}{r^2} \; . \tag{3.13}$$

The components of the rate of strain tensor, equations (3.12), become

$$d_r = -\,2\,\frac{g(\phi)}{r^3} \; ,$$

$$d_\phi = d_\theta = \frac{g(\phi)}{r^3} \; , \tag{3.14}$$

and

$$d_{r\phi} = \frac{g'(\phi)}{2r^3} \; ,$$

upon substitution of relation (3.13). The prime denotes differentiation with respect to ϕ.

The equations of motion are

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\,\frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{1}{r}\,(2\sigma_r - \sigma_\phi - \sigma_\theta + \sigma_{r\phi}\cot\phi) = 0 \; ,$$

and

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\,\frac{\partial \sigma_\phi}{\partial \phi} + \frac{1}{r}\,[(\sigma_\phi - \sigma_\theta)\cot\phi + 3\sigma_{r\phi}] = 0 \; , \tag{3.15}$$

which become, with the substitution (2.2) and noting that

$$S_\theta = -(S_r + S_\phi) \ ,$$

$$\frac{\partial p}{\partial r} = \frac{\partial S_r}{\partial r} + 3\,\frac{S_r}{r} + \frac{1}{r}\,\frac{\partial S_{r\phi}}{\partial \phi} + \frac{1}{r}\,S_{r\phi}\cot\phi \ ,$$

and

$$\frac{\partial p}{\partial \phi} = r\,\frac{\partial S_{r\phi}}{\partial r} + \frac{\partial S_\phi}{\partial \phi} + 3S_{r\phi} \ . \tag{3.16}$$

Elimination of the hydrostatic pressure yields

$$\frac{\partial^2 S_r}{\partial r \partial \phi} + \frac{3}{r}\,\frac{\partial S_r}{\partial \phi} - \frac{\partial^2 S_\phi}{\partial r \partial \phi} + \frac{1}{r}\,\frac{\partial^2 S_{r\phi}}{\partial \phi^2} - r\,\frac{\partial^2 S_{r\phi}}{\partial r^2}$$

$$+ \frac{1}{r}\,\frac{\partial}{\partial \phi}\,(S_{r\phi}\cot\phi) - 4\,\frac{\partial S_{r\phi}}{\partial r} = 0 \ , \tag{3.17}$$

which, again, is entirely in terms of the deviatoric stresses.

Substituting equations (3.14) into the power law gives

$$S_r = -\frac{2^{\frac{5-n}{2}}}{r^{3n}}\,\mu g[12g^2 + (g')^2]^{\frac{n-1}{2}} \ ,$$

$$S_\phi = S_\theta = \frac{2^{\frac{3-n}{2}}}{r^{3n}}\,\mu g[12g^2 + (g')^2]^{\frac{n-1}{2}} \ ,$$

and

$$S_{r\phi} = \frac{2^{\frac{1-n}{2}}}{r^{3n}} \mu g' [12g^2 + (g')^2]^{\frac{n-1}{2}} ,$$

where S_r, S_ϕ, S_θ, and $S_{r\phi}$ are the non-zero components of the deviatoric stress tensor, and the prime denotes differentiation with respect to ϕ.

Letting

$$m(\phi) = \frac{g(\phi)}{V} ,$$

where m is a non-dimensional velocity, and V is a constant to be determined from the volume flow rate, the expressions for the stress deviators become

$$S_r = \frac{2^{\frac{5-n}{2}}}{r^{3n}} \mu V^n \, mG ,$$

$$S_\phi = S_\theta = \frac{2^{\frac{3-n}{2}}}{3n} \mu V^n \, mG , \tag{3.18}$$

and

$$S_{r\phi} = \frac{2^{\frac{1-n}{2}}}{3n} \mu V^n \, m'G ,$$

where

$$G = [12m^2 + (m')^2]^{\frac{n-1}{2}} .$$

Substitution of equations (3.18) into equation (3.17) gives

$$(m'G)'' + (m'G \cot \phi)' + 9n(1-n)\, m'G$$

$$+ 6(3n-2)\,(mG)' = 0 \ , \tag{3.19}$$

and the expression for the hydrostatic pressure becomes

$$p = \frac{2^{\frac{1-n}{2}}}{r^{3n}} \mu V^n \left[\frac{4}{n}(1-n)\, mG - \frac{1}{3n}\{(m'G)' + m'G \cot \phi\}\right] + A. \tag{3.20}$$

Equations (3.19) and (3.20) reduce to those for the quasi-static flow of a Newtonian fluid in a cone for $n = 1$, and De Vries (18) showed that they reduce to the equations obtained by Shield (8) for the rigid-perfectly von Mises solid for $n = 0$. Dividing equation (3.18) by $3n$, integrating, and rearranging, yields

$$\frac{4}{n}(1-n)\, mG - \frac{1}{3n}\{(m'G)' + m'G \cot \phi\}\, m$$

$$= 2mG + 3(1-n)\int_0^{\phi} m'G\, d\phi + A.$$

Substituting this expression in equation (3.20) and evaluating at $n = 0$ gives

$$p = \sqrt{2}\,\mu[2mG + 3\int_0^{\phi} m'G\,d\phi] + E(r) + A\ .$$

This is equivalent to the expression for the pressure found from Shield's equations, since, for n = 0,

$$mG = \pm \frac{1}{2\sqrt{3}} \sqrt{1-(m'G)^2}\ ,$$

and with the substitution

$$\tau = m'G\ ,$$

the expression for the pressure becomes

$$p = \sqrt{2}\,\mu[-\frac{1}{\sqrt{3}}\sqrt{1-\tau^2} + 3\int_0^{\phi} \tau\,d\phi] + E(r) + A\ . \qquad (3.21)$$

The function E(r) can be evaluated since, from equation (3.20),

$$(\frac{\partial p}{\partial r})_{n=0} = \frac{9\sqrt{2}\mu}{r}[(m'G)' + m'G\cot\phi - 12mG],$$

and from equation (3.19)

$$(m'G)' + m'G\cot\phi - 12mG = D\ .$$

From equation (3.21)

$$\left(\frac{\partial p}{\partial r}\right)_{n=0} = E'(r) ,$$

which implies that

$$E(r) = 9\sqrt{2}\,\mu D \ln r + B ,$$

and the expression for the pressure becomes

$$p = \sqrt{2}\,\mu\left[-\frac{1}{\sqrt{3}}\sqrt{1-\tau^2} + 3\int_0^{\phi} \tau\, d\phi + c \ln r\right] + A .$$

The problem is reduced to solving equation (3.19) subject to the boundary conditions

$$m(0) = 1 ,$$

$$m'(0) = 0$$

from symmetry, and the no slip boundary condition

$$m(\alpha) = 0 ,$$

where α is the semi-angle of the cone. Again, in general, this must be done numerically.

De Vries (18) showed that, for $n = \frac{2}{3}$, a closed form solution of equation (3.19) in terms of m'G can be obtained. The solution is

$$m'G = B \sin\phi ,$$

and the shear stress becomes

$$S_{r\phi} = \frac{2^{\frac{1}{6}} \mu V^{\frac{2}{3}}}{r^2} B \sin\phi ,$$

which is used to check the accuracy of the numerical solution.

CHAPTER IV

NUMERICAL METHOD

In the preceding chapter, the governing equations were derived for the flow problems under consideration. These equations can be integrated numerically using Gill's variation of the Runge-Kutta fourth-order method. The problem must first be converted to an initial value problem by assuming a value for the unknown initial condition m"(o). A trial and error procedure to find the value of m"(o) which satisfies the no-slip boundary condition is very costly in computer time. A method developed by Clutter and Smith (19) was adapted to the present problem to systematize the solution of these equations.

4.1 <u>Reformulation as an Initial Value Problem</u>

The Runge-Kutta method can be used to solve a system of first order ordinary differential equations with prescribed initial conditions. The two problems under consideration must first be put in this form.

The plane flow problem, equation (3.9) and the associated boundary conditions, yields the system of first order equations

$$y_1'(\theta) = y_2(\theta) \ ,$$

$$y_2'(\theta) = y_3(\theta) \ ,$$

and

$$y_3'(\theta) = F[y_1(\theta), y_2(\theta), y_3(\theta)] ,$$

where

$$y_1(\theta) = m(\theta) ,$$

$$y_2(\theta) = m'(\theta) ,$$

$$y_3(\theta) = m''(\theta) ,$$

$$F[y_1(\theta), y_2(\theta), y_3(\theta)] = \frac{1}{M+(n-1)(m')^2} \{(4n^2-12n+4)\ m'M$$

$$- [(4n^2-6n+2)\ m + (n-1)m'']\ M'$$

$$- (n-1)(n-3)\ m'(M')^2/4M$$

$$- (n-1)\ m'[4(m')^2 + 4mm'' + (m'')^2]\} ,$$

and

$$M = 4m^2 + (m')^2 .$$

The initial values are

$$y_1(0) = 1 ,$$

$$y_2(0) = 0 \ ,$$

and

$$y_3(0) = \beta \ .$$

β is the unknown value of $m'(0)$ to be determined from the no slip boundary condition.

Similarly, for the axi-symmetric flow problem, equation (3.18) and the associated boundary conditions, gives the system of first order equations

$$y_1'(\phi) = y_2(\phi) \ ,$$

$$y_2'(\phi) = y_3(\phi) \ ,$$

and

$$y_3'(\phi) = F[\phi, y_1(\phi), y_2(\phi), y_3(\phi)] \ ,$$

where

$$F[\phi, y_1(\phi), y_2(\phi), y_2(\phi)] = \frac{1}{M+(n-1)(m')^2} \{[(9n^2-27n+12) \ m'$$

$$- \cot\phi \ m'' + \csc^2\phi \ m'] \ M$$

$$- [(9n^2-15n+6)\, m + (n-1)\, m''$$

$$+ \frac{(n-1)}{2} \cot \phi\, m'\,]\, M'$$

$$- (n-1)(n-3)\, m'(M')^2/4M$$

$$- (n-1)\, m'[12(m')^2 + 12mm'' + (m'')^2]\},$$

$$M = 12m^2 + (m')^2,$$

$$y_1(\phi) = m(\phi)\ ,$$

$$y_2(\phi) = m'(\phi)\ ,$$

and

$$y_3(\phi) = m''(\phi)\ .$$

The initial conditions are

$$y_1(\phi) = 1\ ,$$

$$y_2(\phi) = 0\ ,$$

and

$$y_3(\phi) = \beta \ .$$

Again, β is to be determined from the no slip boundary condition.

4.2 Determination of the Unknown Initial Condition

A purely trial and error procedure to satisfy the no slip boundary condition uses too much computer time. Clutter and Smith (19) encountered the same difficulty in solving the compressible boundary - layer equations for an arbitrary axi-symmetric body. The problems considered here are simpler since the flow is incompressible and thermal effects are neglected. The method can be applied to both the plane and axi-symmetric flow problems.

The procedure is as follows. A value for β is first assumed, and the system of equations integrated. The initial value for β is increased if $m(\alpha)$ is greater than zero, or decreased if $m(\alpha)$ is less than zero. The equations are then integrated again using the new value for β. When both a high and a low value for $m(\alpha)$ is obtained, the two values for β are averaged, and the equations integrated again using this averaged value. This averaging process could be continued until the magnitude of $m(\alpha)$ is as small as desired. This process is speeded up considerably by using a three point interpolation formula. When three solutions, $m_1(\theta)$, $m_2(\theta)$, and $m_3(\theta)$, are obtained such that

$$-k \leq m_1(\alpha) \leq k \ ,$$

$$- k \leq m_2(\alpha) \leq k \ ,$$

and

$$- k \leq m_3(\alpha) \leq k \ ,$$

where k is a prescribed limit, and two of $m_1(\alpha)$, $m_2(\alpha)$, and $m_3(\alpha)$ are greater than zero and the third less than zero, or conversly, they are combined by an interpolation formula which automatically satisfies all the boundary conditions. The interpolated solution can be checked by integrating the equations using the interpolated value for β. Clutter and Smith found the computer time required using this method to be about one fourth that required if the averaging process was used.

The interpolated solution is found by substituting the three solutions in the formulas

$$m(\theta) = A_1 m_1(\theta) + A_2 m_2(\theta) + A_3 m_3(\theta) \ ,$$

$$m'(\theta) = A_1 m_1'(\theta) + A_2 m_2'(\theta) + A_3 m_3'(\theta) \ ,$$

and

$$m''(\theta) = A_1 m_1''(\theta) + A_2 m_2''(\theta) + A_3 m_3''(\theta) \ ,$$

where

$$A_1 = \frac{m_2(\alpha)\ m_3(\alpha)}{[m_1(\alpha) - m_2(\alpha)][m_1(\alpha) - m_3(\alpha)]} ,$$

$$A_2 = \frac{m_1(\alpha)\ m_3(\alpha)}{[m_2(\alpha) - m_1(\alpha)][m_2(\alpha) - m_3(\alpha)]} ,$$

and

$$A_3 = \frac{m_1(\alpha)\ m_2(\alpha)}{[m_3(\alpha) - m_1(\alpha)][m_3(\alpha) - m_2(\alpha)]} ,$$

By substitution, it is easy to see that all the boundary conditions are satisfied by the interpolated solution.

The other quantities of interest, pressure and shear stress, can be found in the same way.

The value of k must be chosen to give the required accuracy. A value for k of 0.2 was found to give solutions which agreed to four decimal places with the solution found by integrating the equations using the interpolated value for β. Also, good agreement with the known solutions discussed in the preceding chapters was obtained. Some experimentation was necessary to determine the value of k which gave sufficiently accurate solutions using a minimum of computer time.

The value of β assumed initially is important in reducing computer time. Since the solutions for both plane flow in a converging channel and axi-symmetric flow in a cone are known for a Newtonian fluid, for which n = 1, and it was

desired to solve the equations for a series of values for n less than one, the known value for β for n = 1 was used as the first trial value for β for n = 0.9. A series of solutions for n = 1 down to n = 0.01, in steps of 0.1, was obtained in this way. Only four trials were necessary to obtain the three solutions needed for use in the interpolation formula for each value of n.

Some trial solutions and a comparison of the interpolated solution to the solution found by integrating the equations using the interpolated value for β, for plane flow, a semi-angle of fifteen degrees, and n = 0.5, are presented in the next chapter.

4.3 The Special Case: n = 0

No difficulty is encountered in obtaining solutions for arbitrarily small values of n. However, the computer time needed increases with decreasing n, since smaller step sizes are required. When n = 0 the situation is different. The velocity derivative is infinite at the boundary, and therefore the numerical solution can not be extended right to the boundary. Two approaches are possible:

1. The zero velocity boundary condition can be satisfied at a point arbitrarily close to the boundary.

2. The values of β satisfying the zero velocity boundary condition for $n \leq 1$ can be extrapolated to zero, and the extrapolated value used to integrate the equations for n = 0.

The first approach is unattractive since too much computer time is required to get a solution sufficiently close to the

boundary. The second approach was taken by De Vries (20).

A shear stress boundary condition can also be imposed and the same numerical methods discussed above employed. Good agreement with Nadai's solution for the rigid-perfectly plastic Von Mises solid was obtained in this way for a semi-angle of 15 degrees.

CHAPTER V

RESULTS

The results of the numerical analyses are presented in this chapter. First, some tables are given to demonstrate the merit of the numerical method, and second, the solutions obtained are given in the form of graphs for semi-angles of 5, 10, and 15 degrees, and various values of n.

Table 5.1 gives the trial values for β used to obtain the solution to the plane flow problem for a semi-angle of 15 degrees and n equal to 0.5. The first trial value is the value of β which satisfies the no slip boundary condition for n equal to 0.6. The next three trials gave the solutions required for substitution in the interpolation formula. As a check on the validity of the interpolated solution, the equation was integrated using the interpolated value for β. The solutions obtained by the two methods are compared in table 5.2. The solutions agree to four significant figures. A further check on the method is to compare the numerical solutions obtained for the axi-symmetric flow problem for n equal to $\frac{2}{3}$, and for the plane flow problem for n equal to $\frac{1}{2}$, to the analytic solutions. This is done in table 5.3, and again there is satisfactory agreement. These results show the validity of the method. Closer agreement could be obtained by decreasing the value of k, with a consequent increase in computer time.

Figures 5.1 through 5.9 present the results obtained

for plane flow in a converging channel for semi-angles of 5, 10 and 15°, and figures 5.10 through 5.15 the results for axi-symmetric flow in a cone, for semi-angles of 5 and 10°, and various values of n.

Figures 5.16 through 5.19 give the results for the plane flow of a rigid-perfectly plastic Von Mises solid in a converging channel with a shear stress boundary condition. The results agree with Nadai's solution.

TABLE 5.1

TRIAL SOLUTIONS FOR PLANE FLOW

$n = 0.5 \quad \alpha = 15^{\circ}$

Trial	β	$v_r(\alpha)$
1	-19.9814	-0.245150
2	-16.9814	+0.073443
3	-18.4814	-0.078910
4	-17.6909	+0.003114
Interpolated	-17.7211	0.000000

TABLE 5.2

COMPARISON OF INTERPOLATED AND
INTEGRATED SOLUTIONS FOR PLANE FLOW

$\alpha = 15^\circ$

θ Degrees	$m(\theta)$ Interpolated Solution	$m(\theta)$ Integrated Solution
1	1.000000	1.000000
3	0.975194	0.975193
6	0.893993	0.893990
9	0.732847	0.732843
12	0.450761	0.450757
15	0.000000	0.000003

θ Degrees	$p(\theta)$ Interpolated Solution	$p(\theta)$ Integrated Solution
1	0.000000	0.000000
3	0.130321	0.130323
6	0.501925	0.501933
9	1.028350	1.028397
12	1.599365	1.599467
15	2.189448	2.189571

θ Degrees	$S_{r\theta}$ Interpolated Solution	$S_{r\theta}$ Integrated Solution
1	0.000000	0.000000
3	0.779888	0.779913
6	1.557638	1.557689
9	2.331119	2.331195
12	3.098210	3.098312
15	3.856809	3.856936

TABLE 5.3

COMPARISON OF THEORETICAL AND

NUMERICAL SOLUTIONS

$\alpha = 15^\circ$

Axi-Symmetric Flow (n=2/3)

ϕ	$S_{r\phi}$	
Degrees	Theoretical	Numerical
0	0	
1	0.32179	0.32180
2	0.64349	0.64350
3	0.96498	0.96500
4	1.28619	1.28621
5	1.60700	1.60703
6	1.92732	1.92736
7	2.24706	2.24711
8	2.56611	2.56616
9	2.88438	2.88444
10	3.20177	3.20184
11	3.51818	3.51826
12	3.83353	3.83361
13	4.14770	4.14779
14	4.46062	4.46071
15	4.77217	4.77227

TABLE 5.3 (CONTINUED)

COMPARISON OF THEORETICAL AND NUMERICAL SOLUTIONS

$\alpha = 15$

Plane Flow (n=1/2)

θ	$S_{r\theta}$	
Degrees	Theoretical	Numerical
0	0	0
1	0.26008	0.26007
2	0.52007	0.52006
3	0.77991	0.77989
4	1.03952	1.03948
5	1.29880	1.29876
6	1.55769	1.55764
7	1.81610	1.81604
8	2.07397	2.07390
9	2.33120	2.33112
10	2.58772	2.58763
11	2.84345	2.84335
12	3.09831	3.09821
13	3.35223	3.35212
14	3.60513	3.60502
15	3.85694	3.85681

RESULTS

FOR

PLANE FLOW

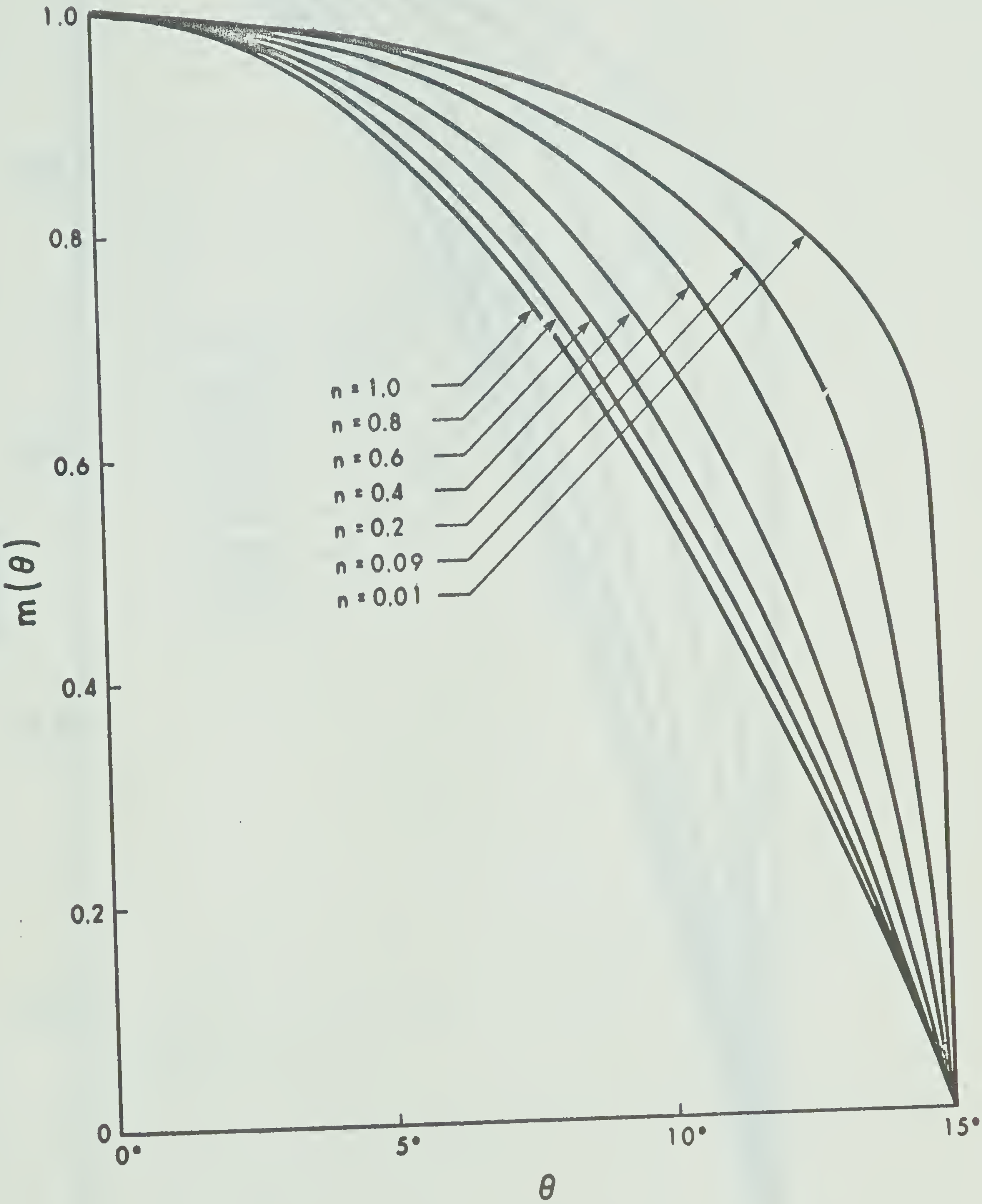

FIGURE 5.1

$m(\theta)$ VERSUS θ: $\alpha = 15^\circ$

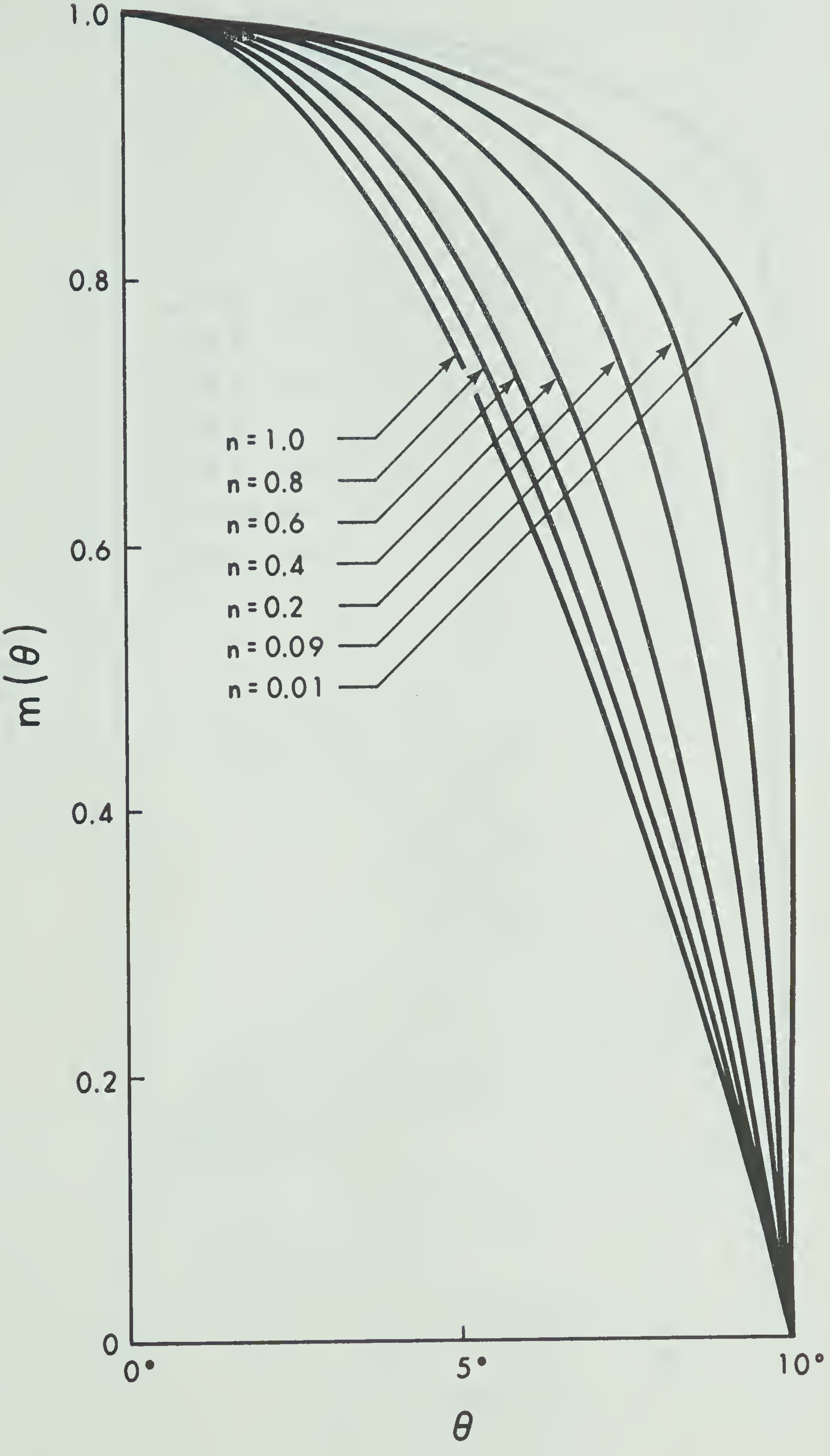

FIGURE 5.2

m (θ) VERSUS θ: $\alpha = 10^{\circ}$

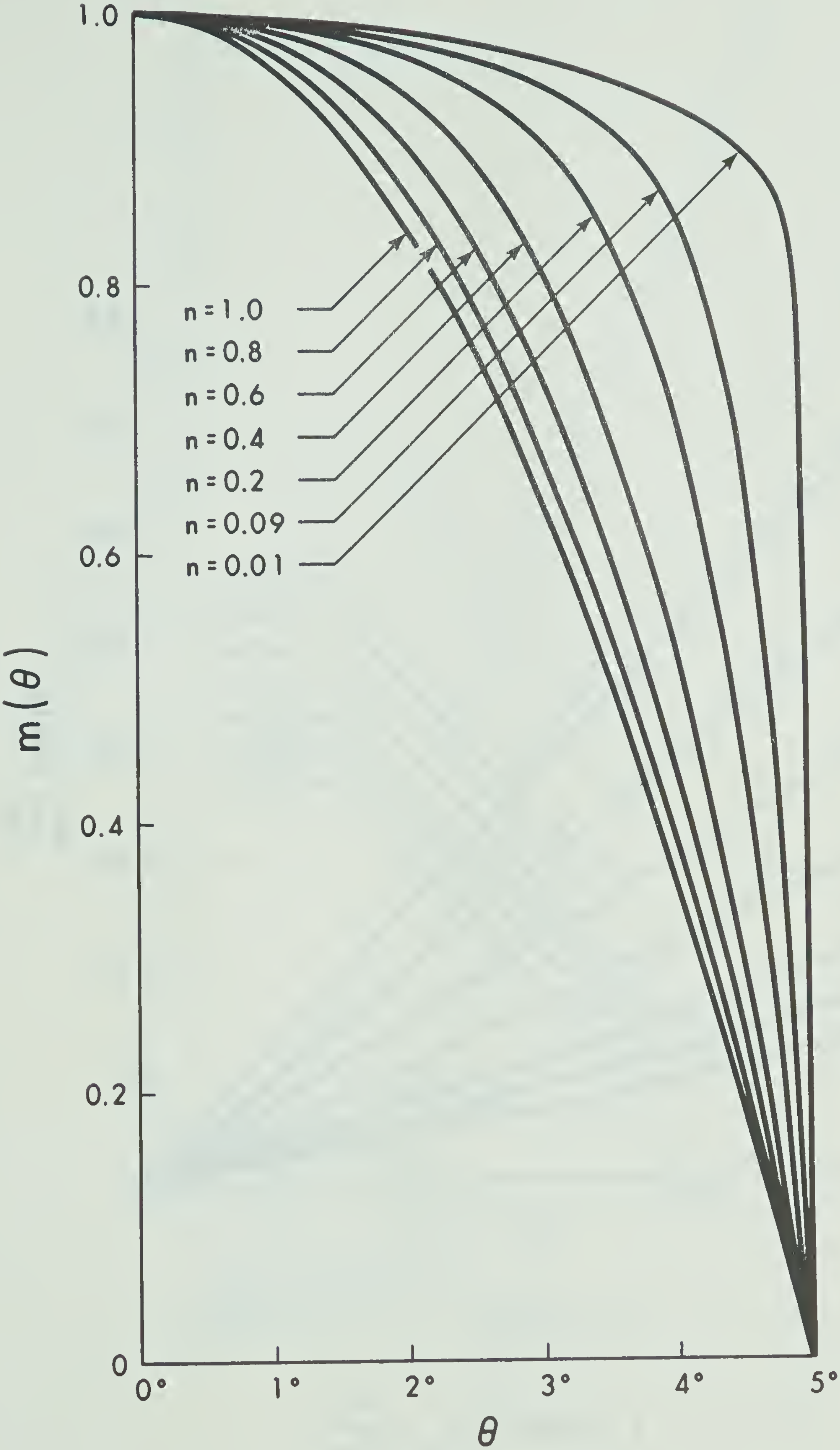

FIGURE 5.3

m (θ) VERSUS θ: α = 5°

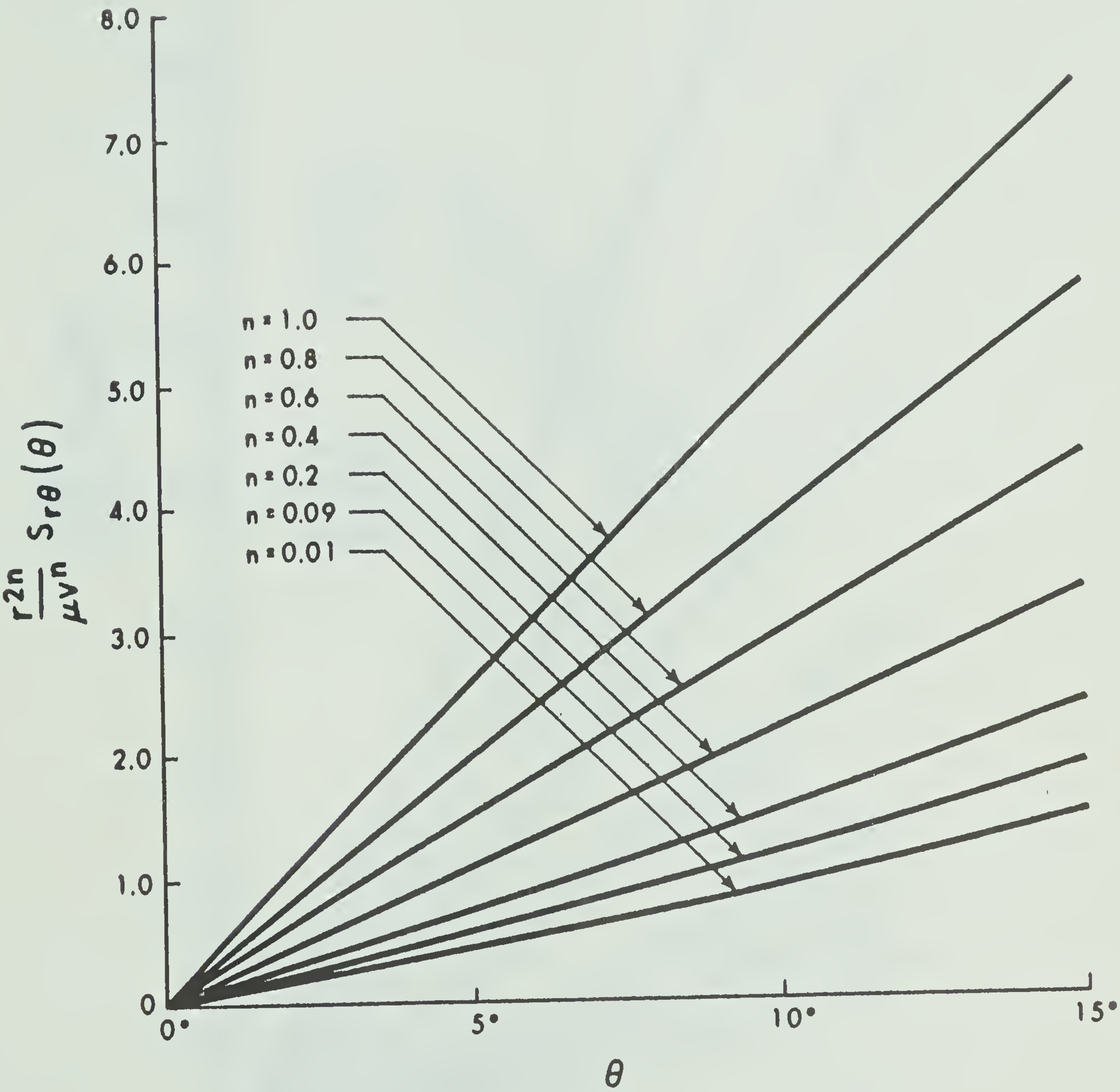

FIGURE 5.4

$\frac{r^{2n}}{\mu V^n} S_{r\theta}(\theta)$ VERSUS θ: $\alpha = 15°$

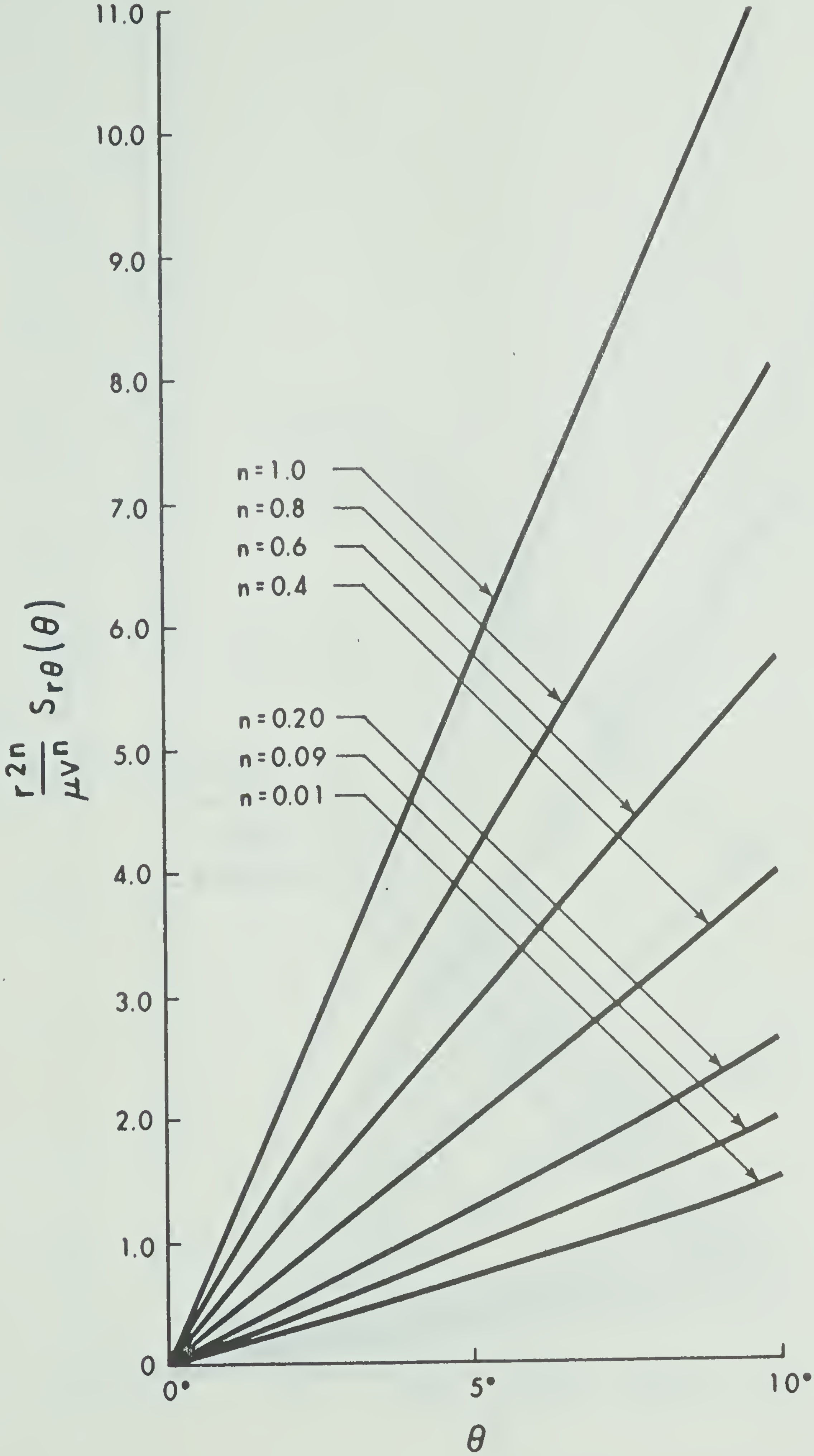

FIGURE 5.5

$\frac{r^{2n}}{\mu V^n} S_{r\theta}(\theta)$ VERSUS θ: $\alpha = 10$

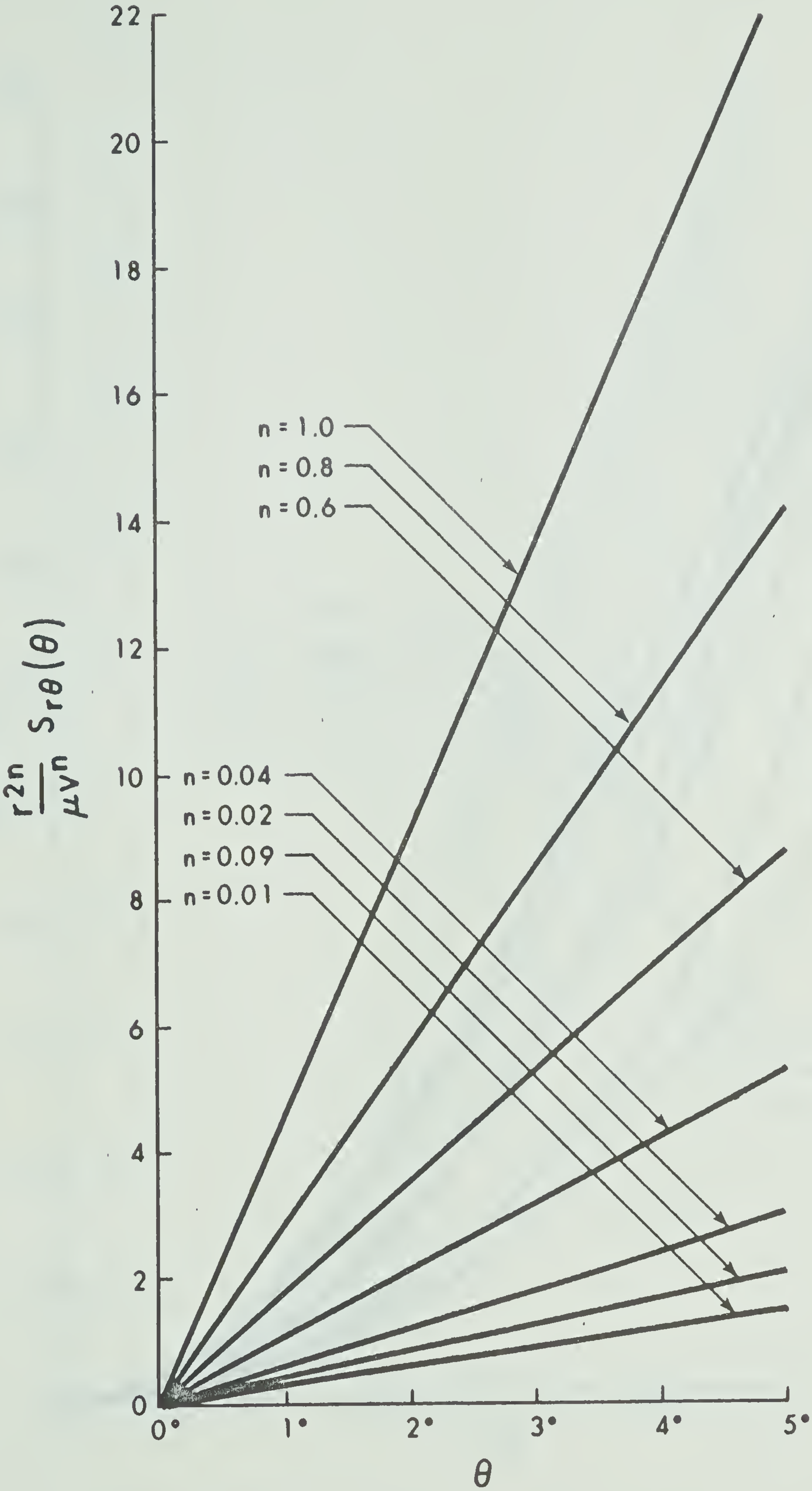

FIGURE 5.6

$\frac{r^{2n}}{\mu v^{n}} S_{r\theta}(\theta)$ VERSUS θ: $\alpha = 5^{\circ}$

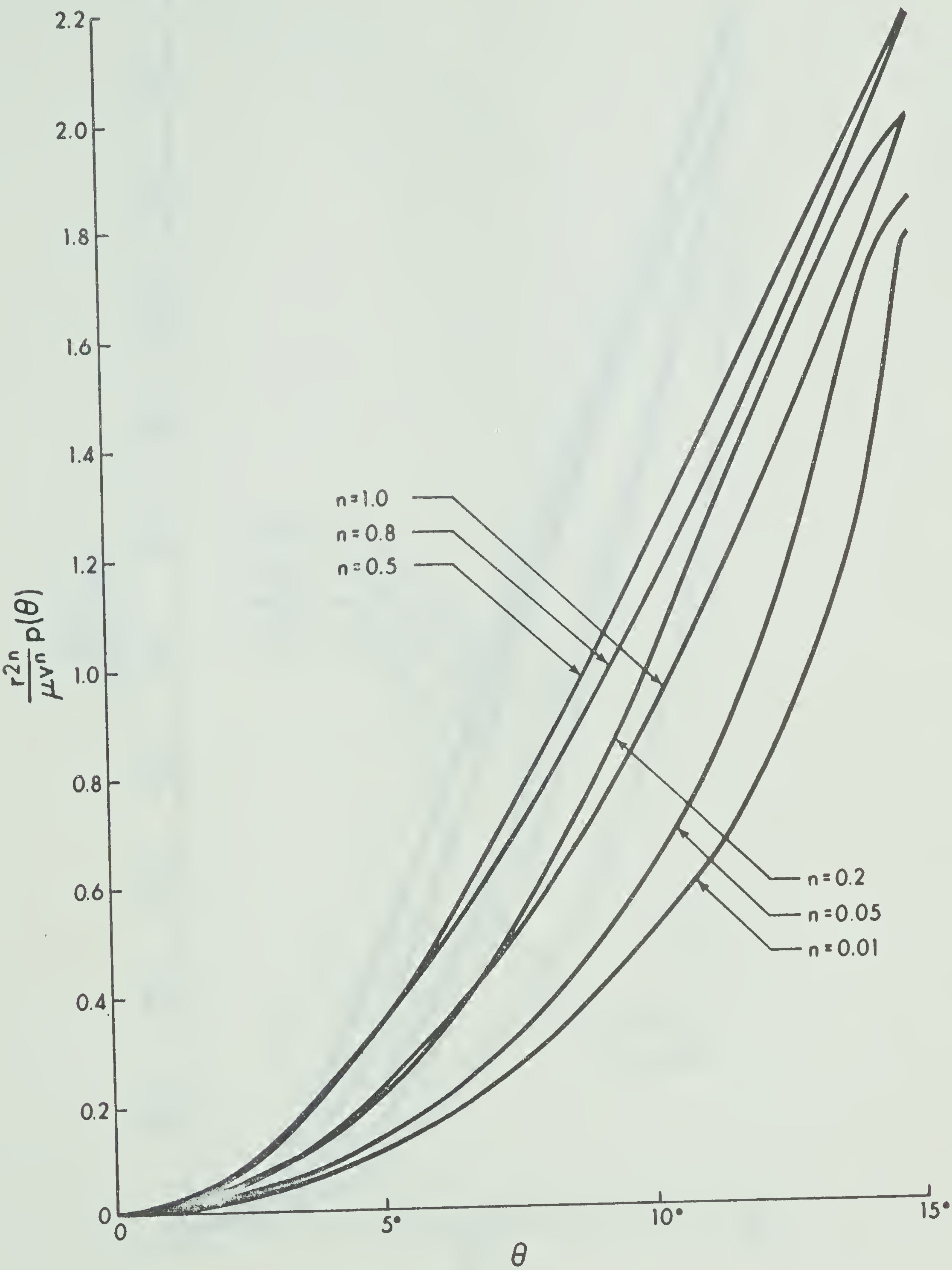

FIGURE 5.7

$\frac{r^{2n}}{\mu V^n} p(\theta)$ VERSUS θ: $\alpha = 15^\circ$

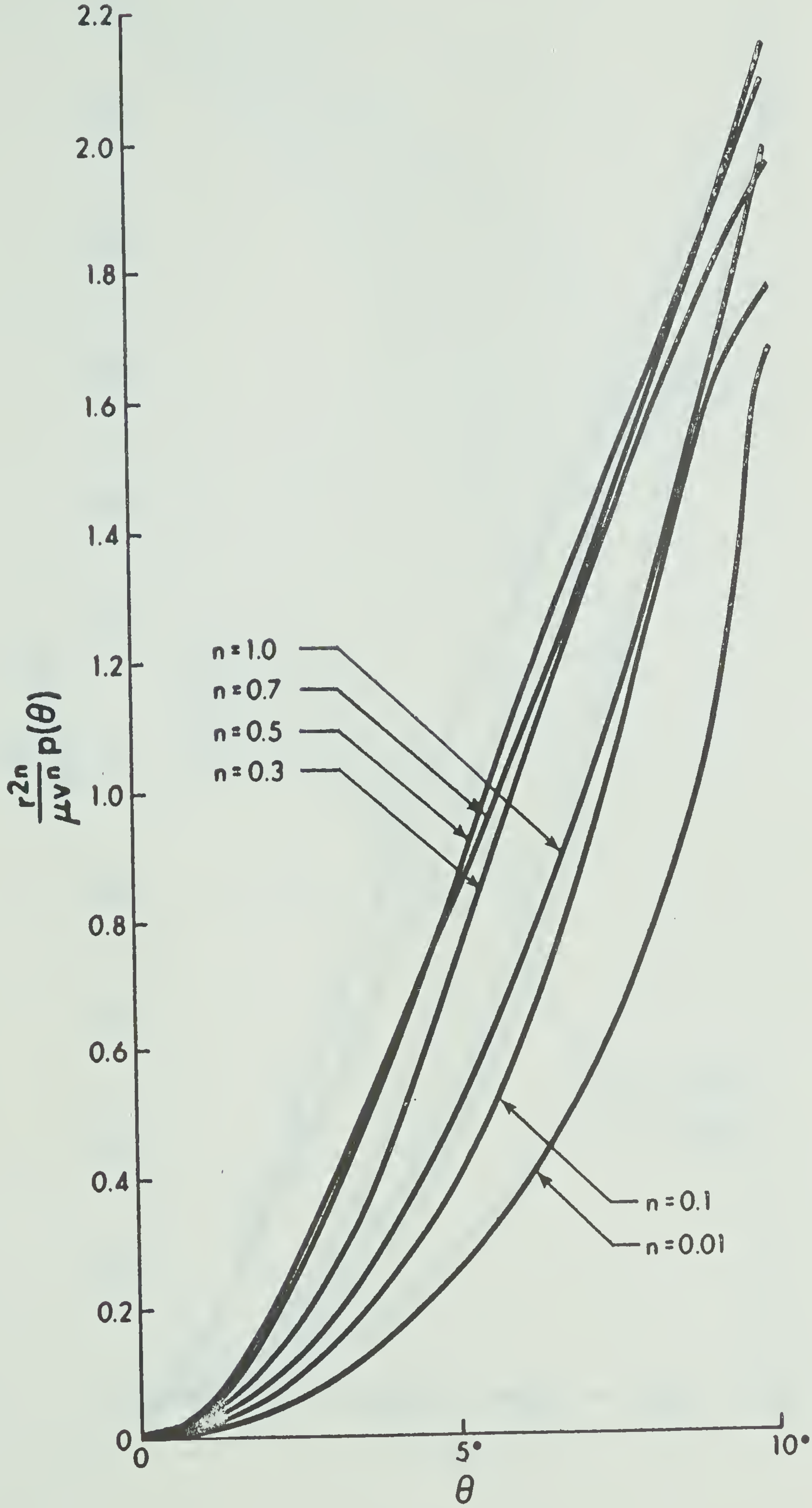

FIGURE 5.8

$\frac{r^{2n}}{\mu V^n} P(\theta)$ VERSUS θ; $\alpha = 10^\circ$

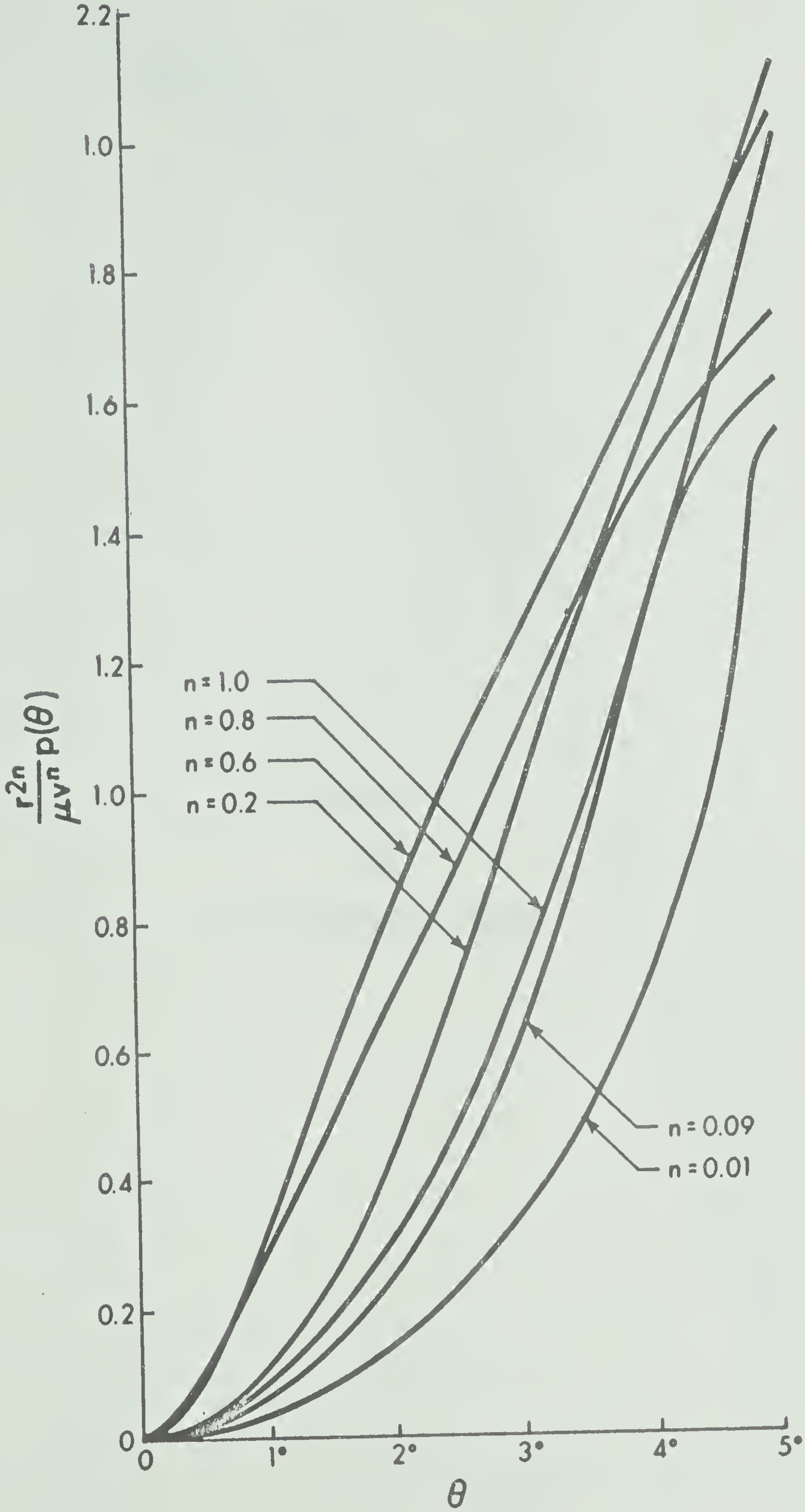

FIGURE 5.9

$\frac{r^{2n}}{\mu V^n} p(\theta)$ VERSUS θ: $\alpha = 5°$

RESULTS

FOR

AXI-SYMMETRIC FLOW

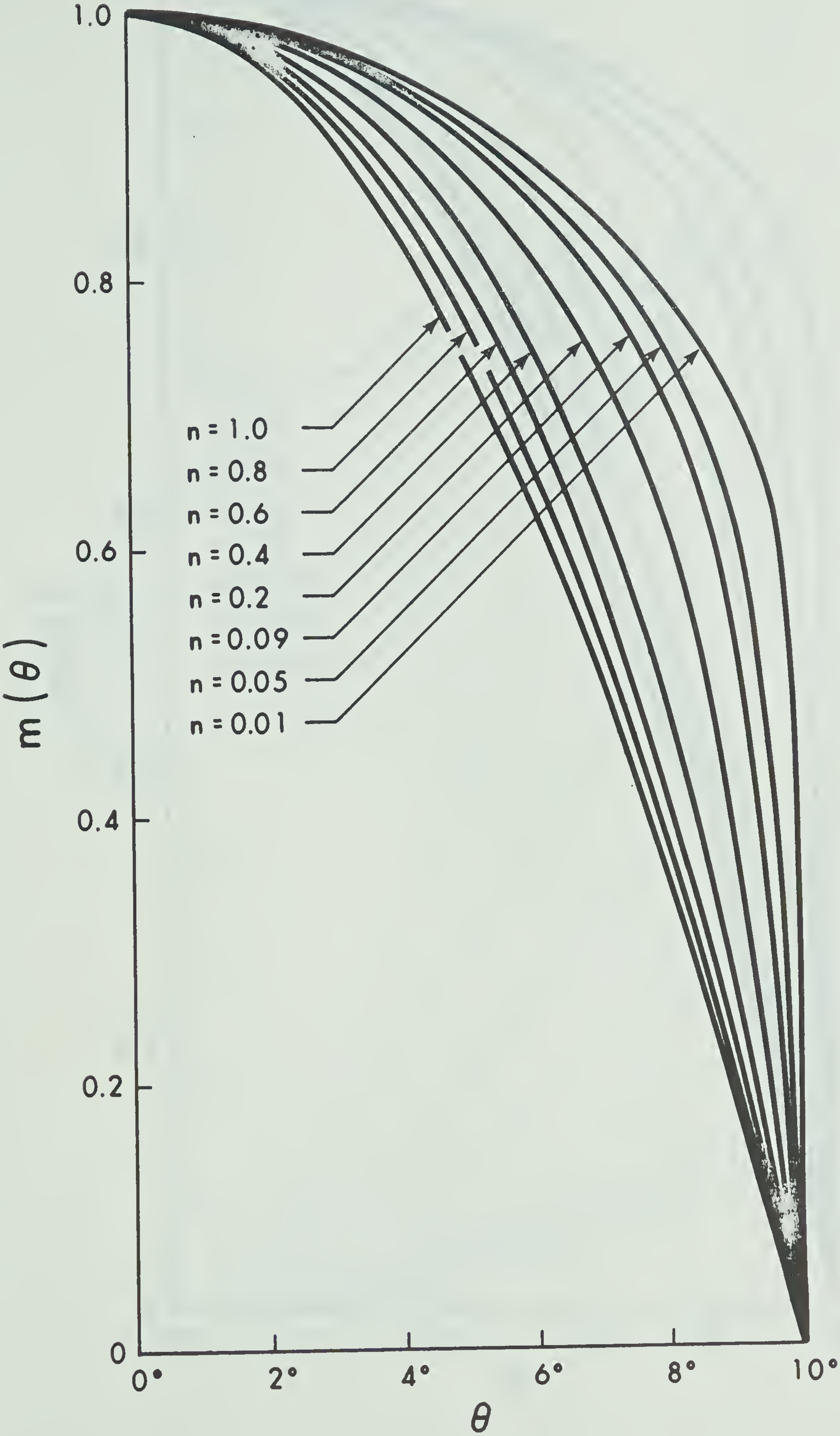

FIGURE 5.10

$m(\theta)$ VERSUS θ: $\alpha = 10^{\circ}$

FIGURE 5.11

$m(\theta)$ VERSUS θ: $\alpha = 5^\circ$

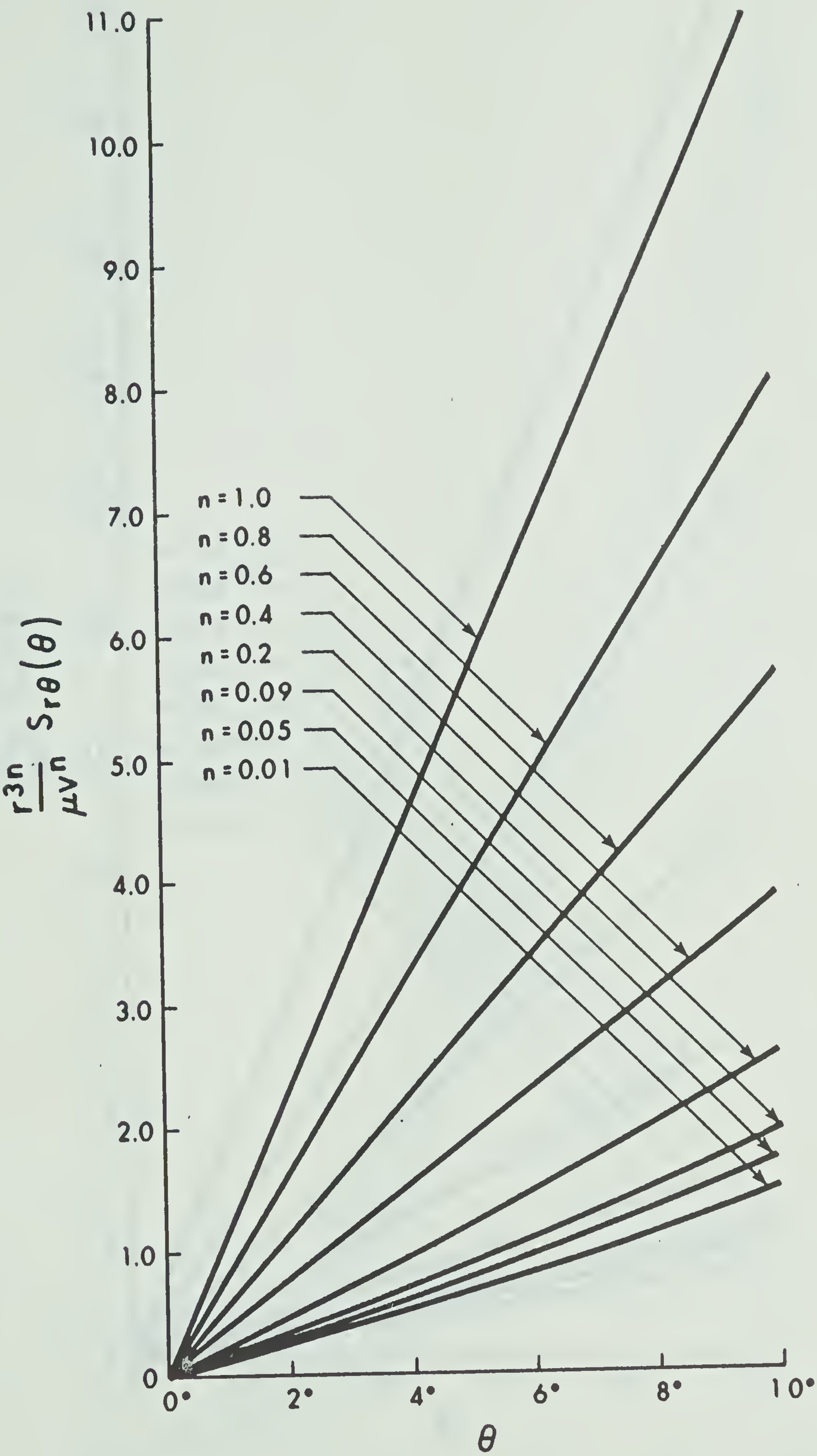

FIGURE 5.12

$\frac{r^{3n}}{\mu V^n} S_{r\theta}(\theta)$ VERSUS θ: $\alpha = 10^\circ$

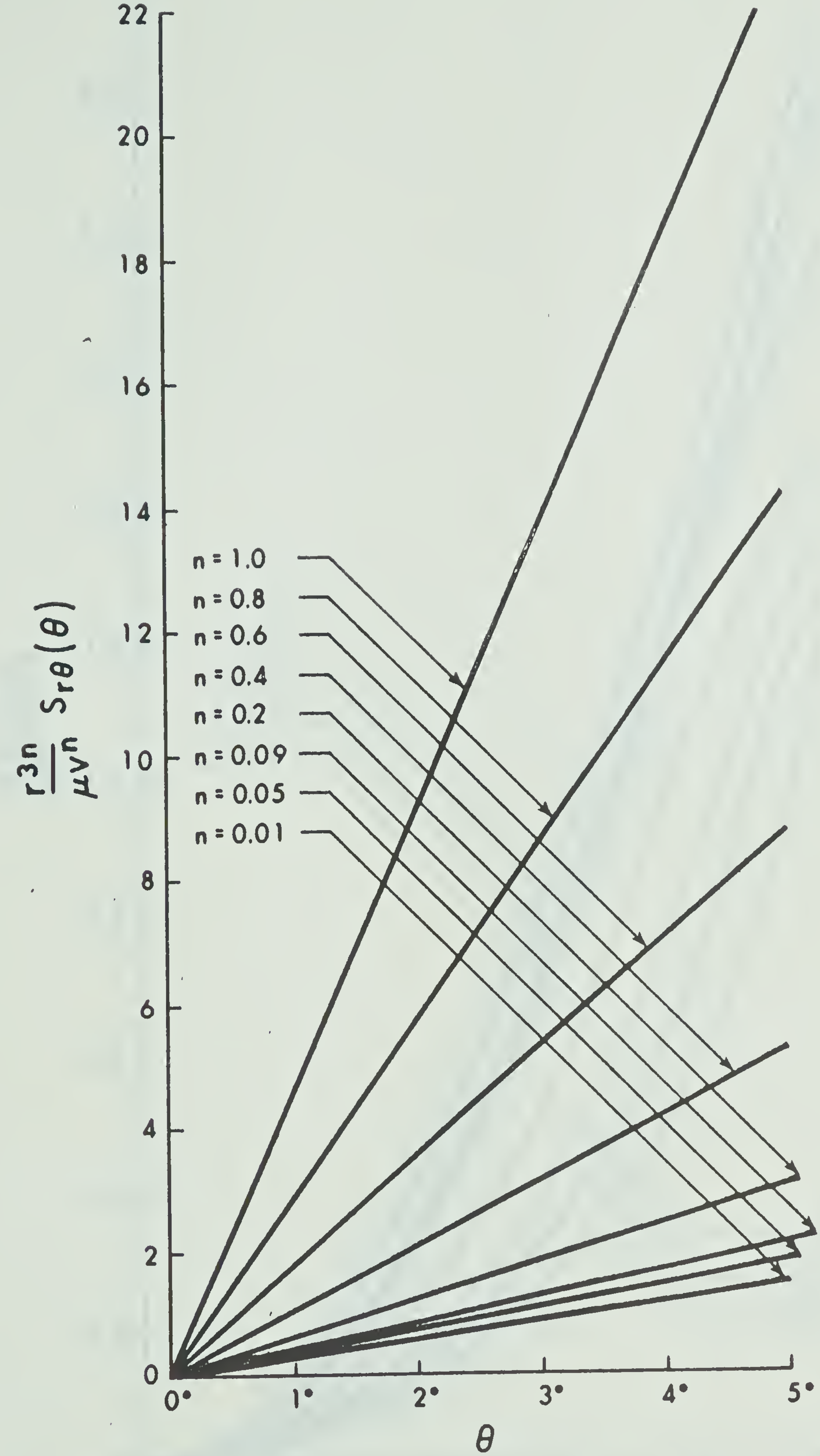

FIGURE 5.13

$\frac{r^{3n}}{\mu V^n} S_{r\theta}(\theta)$ VERSUS θ: $\alpha = 5°$

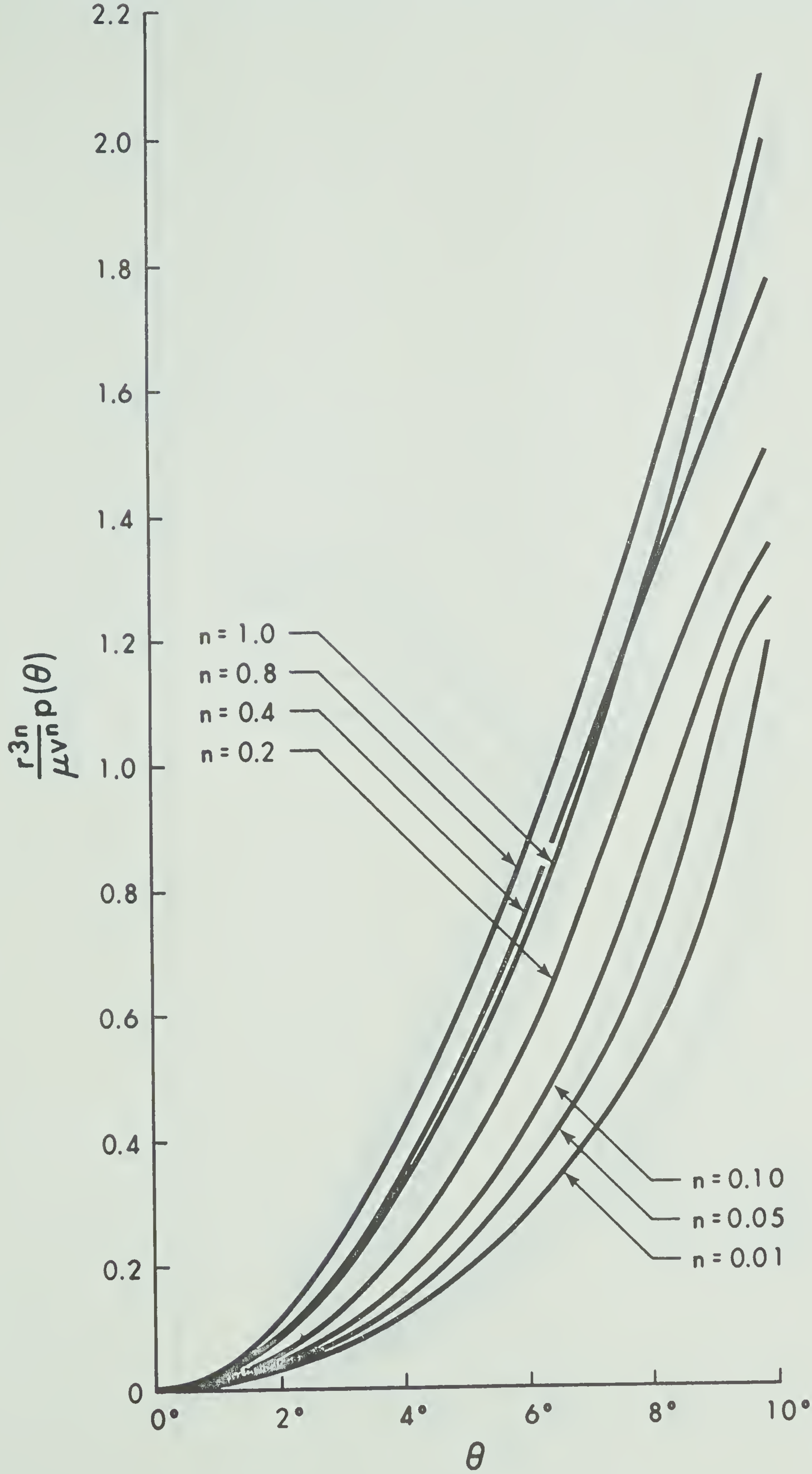

FIGURE 5.14

$\frac{r^{3n}}{\mu V^n}$ p(θ) VERSUS θ: α = 10°

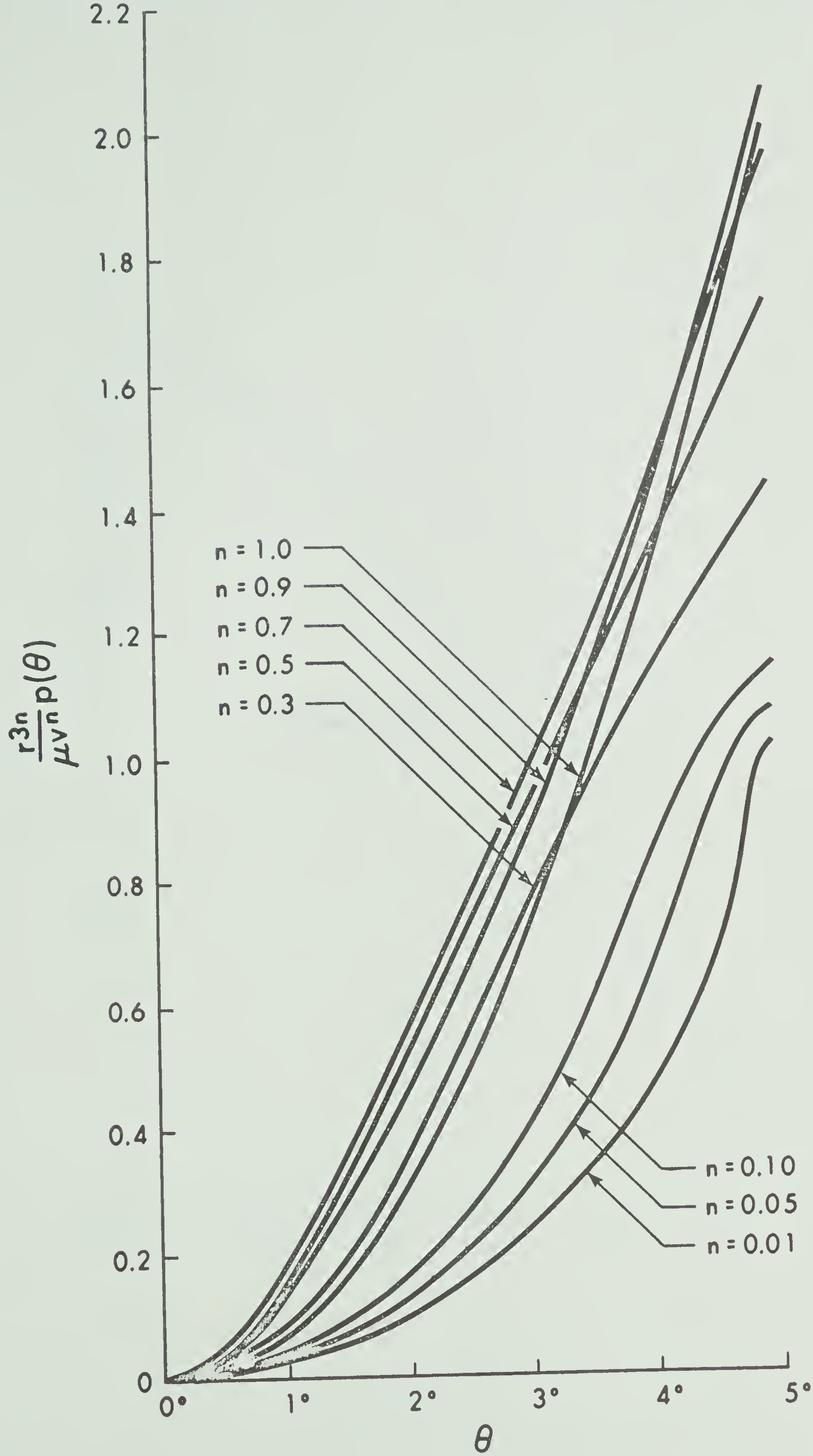

FIGURE 5.15

$\frac{r^{3n}}{\mu V^n}$ p(θ) VERSUS θ: α = 5°

PLANE FLOW

$n = 0$

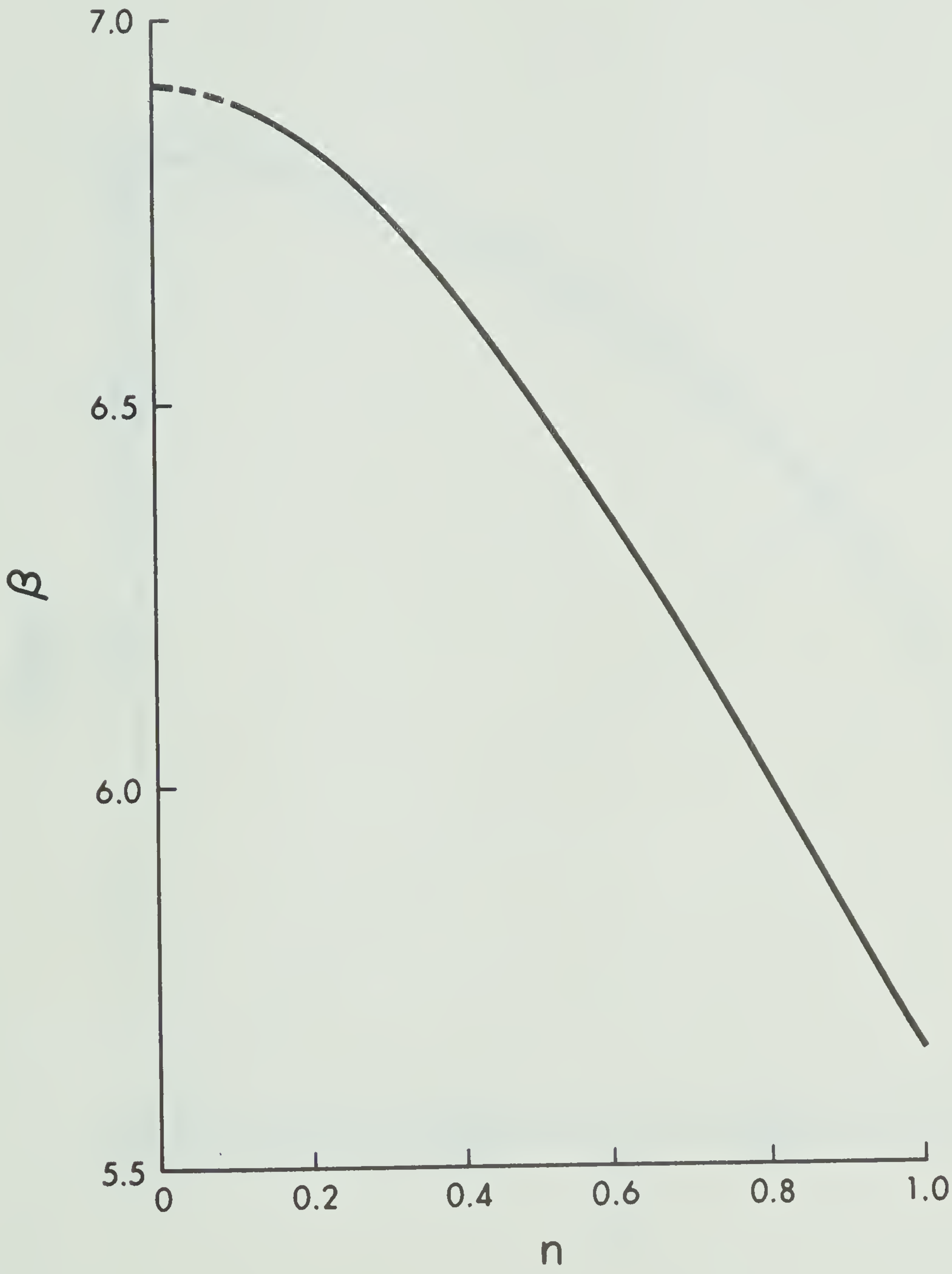

FIGURE 5.16

SHEAR STRESS BOUNDARY CONDITION

β VERSUS n: $\alpha = 15^\circ$, $n = 0$

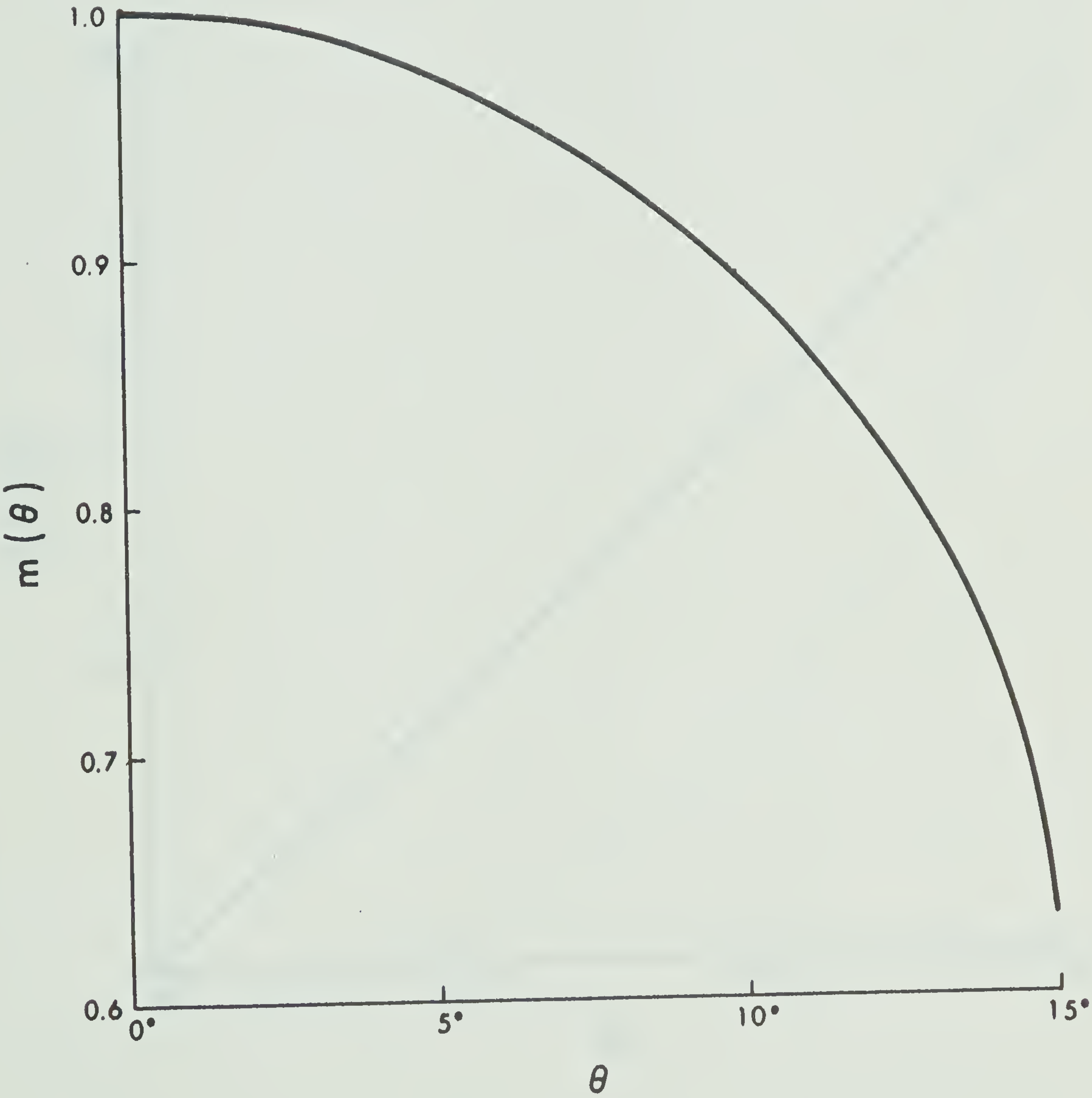

FIGURE 5.17

SHEAR STRESS BOUNDARY CONDITION

$m(\theta)$ VERSUS θ: $\alpha = 15^{\circ}$, $n = 0$

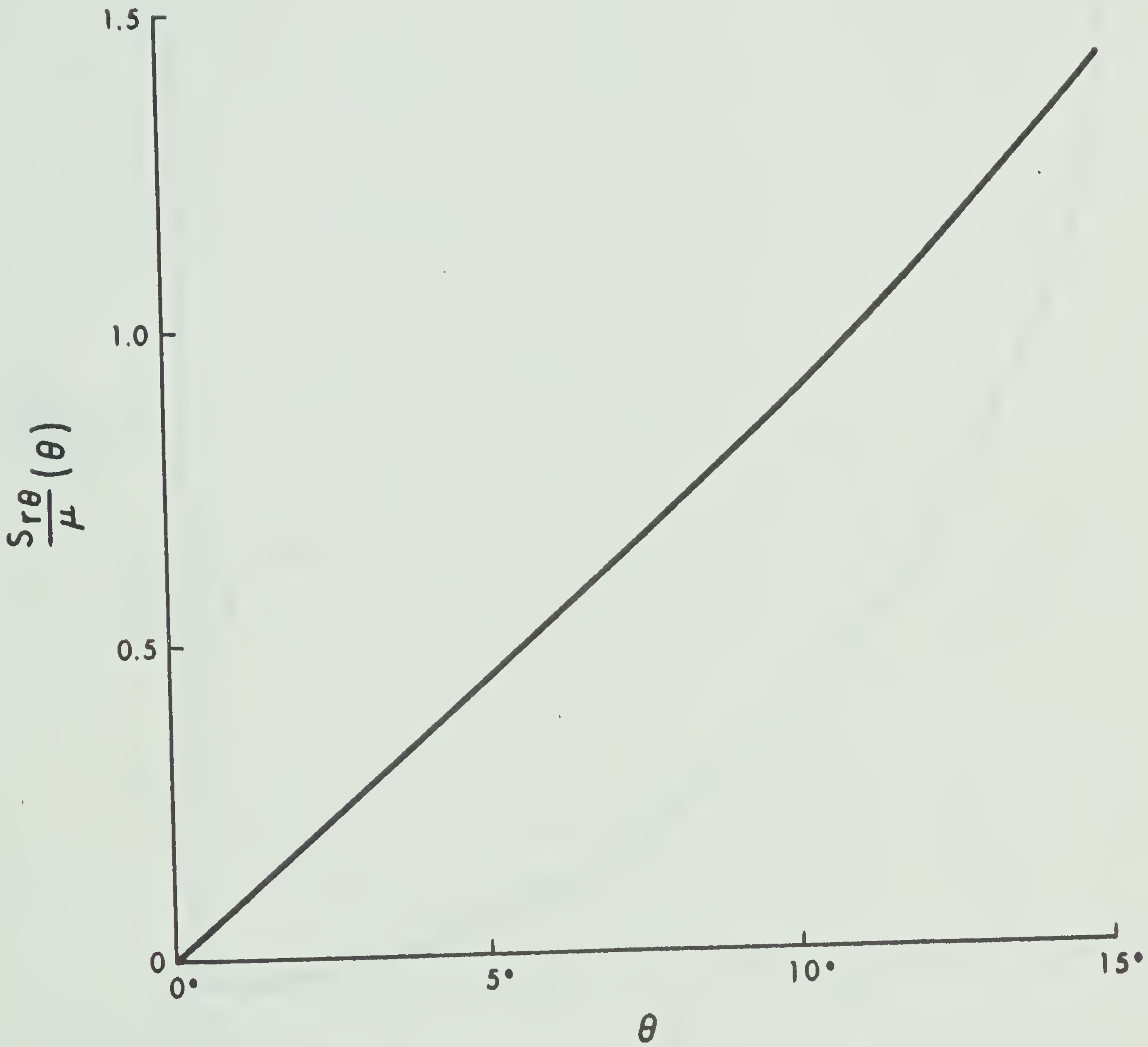

FIGURE 5.18

SHEAR STRESS BOUNDARY CONDITION

$\frac{S_{r\theta}(\theta)}{\mu}$ VERSUS θ: $\alpha = 15^\circ$, $n = 0$

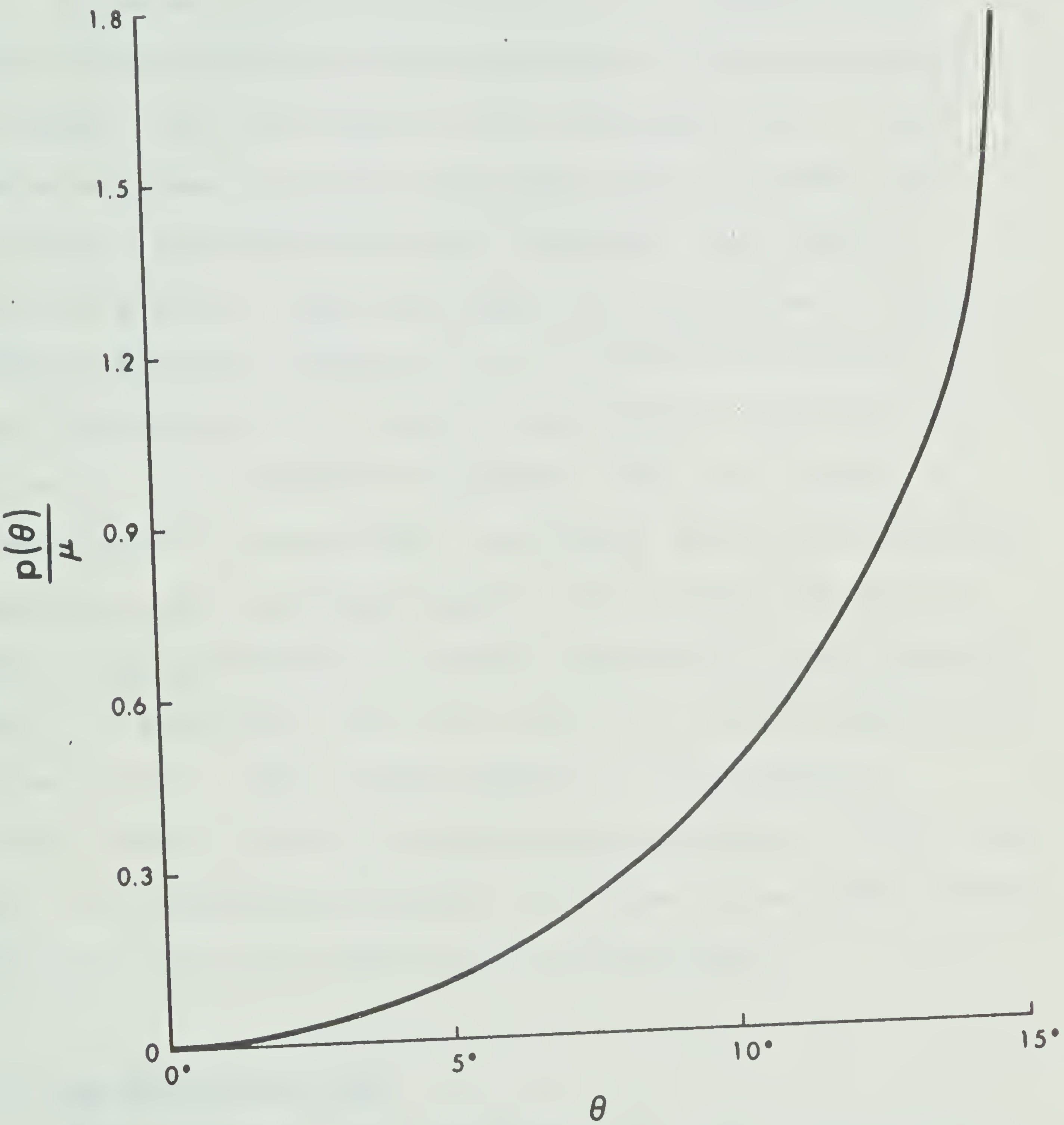

FIGURE 5.19

SHEAR STRESS BOUNDARY CONDITION

$\frac{p(\theta)}{\mu}$ VERSUS θ: $\alpha = 15^\circ$, $n = 0$

CHAPTER VI

INLET EFFECTS

In order to obtain solutions to the pseudo-plastic flow problems considered in this thesis, it was necessary to assume, from the outset, fully developed flow. That the streamlines for fully developed flow are radial was verified, since solutions were obtained. The velocity profiles given by these solutions are established at some distance from the entrance, but to find this distance it would be necessary to consider the complete equations of motion and the continuity equation. This is a system of three coupled, second order, non-linear partial differential equations which are very complex. An attempt was made to solve a simplified form of these equations for axi-symmetric flow of a Newtonian fluid in a cone by a method suggested by Sutterby (21). This method appears to be inadequate. Another method, based on variational principles, offers some hope for obtaining approximate solutions for entrance effects. This is an area for future work in this field.

6.1 The Newtonian Fluid

Both Sutterby (21) and Tanner (13) have investigated entrance effects for the flow of a Newtonian fluid in a cone. Sutterby applied a numerical technique, while Tanner used analytic methods. In both cases the results obtained appear to be neither conclusive nor very satisfactory.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

For small semi-angles, Sutterby assumed that the tangential velocity would be small, and reduced the equations of motion and continuity to

$$\rho v_r \frac{\partial v_r}{\partial r} = - \frac{dp}{dr} + \frac{\mu}{r^2} \left[\frac{1}{\theta} \frac{\partial}{\partial \theta} \left(\theta \frac{\partial v_r}{\partial \theta}\right)\right] ,$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{\theta} \frac{\partial}{\partial \theta} (\theta v_\theta) = 0 ,$$

and

$$Q = 2\pi r^2 \int_0^\alpha v_r \, \theta \, d\theta .$$

For entrance conditions a constant pressure, a zero tangential velocity, and a parabolic profile for the radial velocity,

$$v_r = V[1 - (\frac{\phi}{\alpha})^2] ,$$

were assumed. For small semi-angles this is very close to the radial flow solution given by equation (1.6). Using finite differences, the equations were solved for various values of the Reynolds number. The results showed that, as the Reynolds number approaches zero, the solution approaches the quasi-static radial flow solution, and for large Reynolds numbers it approaches the ideal inviscid radial flow solution. However, a solution for a Reynolds number of

[illegible]

$$[illegible]$$

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

zero was not given; it was assumed that the known radial flow solution would be the solution, regardless of conditions at the entrance and the neglect of the variation in the pressure with θ. This method was found to yield results for the parabolic inlet velocity profile, and for an arbitrarily small Reynolds number, which appeared to be correct. However, for zero Reynolds number, the solution did not approach the radial flow solution at some distance from the entrance as was expected. Inclusion of the variation of pressure in the θ direction had an insignificant effect on the results. A more severe test of the method was tried using the inlet velocity profile

$$v_r = V[1 - (\frac{\theta}{\alpha})^{10}] .$$

This change in the inlet conditions was not reflected in the results. This led to the conclusion that the equations used by Sutterby are over-simplified, and are inadequate to describe the flow, at least in the low Reynolds number regime, and that reasonable solutions were obtained only because of the choice of a parabolic inlet velocity profile. To find the distance from the inlet at which the flow is fully developed, a much more complicated form of the equations must be solved.

Tanner (13) attempted to find the effect of entrance conditions by introducing Stokes' stream function ψ in the inertialess equations of motion in spherical coordinates, which yields

$$E^2\psi = \sin\theta \frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial r}\right) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right)]^2 \psi = 0 ,$$

where

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} ,$$

and

$$v_\theta = -\frac{1}{r \sin\theta}\frac{\partial\psi}{\partial r} .$$

The solution

$$\psi = \sum_m r^m f_m(X),$$

where

$$X = \cos\theta ,$$

was assumed, and solutions for the f_m in terms of Legendre functions were obtained. The values of m which satisfy the boundary conditions were found to be complex. These values of m were used to find the distance at which inlet effects would no longer be felt. For semi-angles smaller than thirty degrees, Tanner found this distance to be approximately equal to the radius of the cone at the inlet, but the manner in which this conclusion was reached is not clear.

The problem of entrance effects for the quasi-static

flow of a Newtonian fluid in a cone cannot be considered solved. Experimental work and refined numerical analysis are needed to settle the question.

6.2 Variational Principles

The problem of entrance effects for a Newtonian fluid is complex, and satisfactory solutions have not yet been obtained for the flows considered in this thesis. The equations describing the flow of non-Newtonian materials, and, in particular, power law materials, are much more complicated, and there seems to be little hope for obtaining exact solutions for entrance effects.

Bird (22) showed that the equations of motion and continuity for the quasi-static flow of an incompressible power law material are equivalent to the Euler-Lagrange equations for the integral

$$\iiint F \, dv \, , \tag{6.1}$$

where

$$F = \frac{1}{2} \eta \; \frac{I_2}{(n+1)} - (p + \rho\phi) \; I \; ,$$

$$\eta = 2\mu(2I_2)^{\frac{n-1}{2}} \; ,$$

ϕ is the potential function for the body forces, and I_1 and

I_2 are the first and second invariants of the rate of strain tensor. The velocity must be specified over the entire surface of the volume over which (6.1) is taken.

With this variational principle approximate solutions may be obtainable. If a velocity distribution, containing arbitrary constants, that satisfies the no-slip boundary condition and gives the prescribed velocity profile at the inlet, and fully developed flow at some distance down stream, can be constructed, then the arbitrary constants may be chosen to minimize (6.1). Even for simple velocity distributions the expression for F will be very complicated, and even if a solution can be found in this way, there is no way of knowing how close the approximate solution is to the exact solution. Convergence could be checked by taking more arbitrary constants in the assumed velocity distribution. This variational principle also opens the possibility of using the finite element technique. This is an area for further investigation.

CHAPTER VII

CONCLUSION

In this thesis the power law was considered as an empirical approximation for the constitutive equations of pseudo-plastic materials for a certain range of flow rate. Experimental results are needed to compare with the theoretical results obtained in order to check the validity of the power law. The assumption that the streamlines are radial for fully developed flow was justified since solutions were obtained, but the problem of entrance and exit effects still remains unsolved. More work needs to be done in this area.

The numerical technique developed in this thesis can be adapted to most two point boundary value problems for ordinary differential equations. It appears to be more straight forward and more universally applicable than other techniques, such as quasi-linearization and invariant imbedding, and is a great deal more economical in computer time than the usual trial and error procedure. Hence, it should prove of value in solving other two point boundary value problems.

BIBLIOGRAPHY

1. A. G. Fredrickson, "Principles and Applications of Rheology", Prentice-Hall, Inc., Englewood Cliffs, N. J., 1964, p. 3.

2. R. Aris, "Vectors, Tensors, and the Basic Equations of Fluid Mechanics", Prentice-Hall, Inc., Englewood Cliffs, N. J., 1962, pp. 133-192.

3. R. S. Brodkey, "The Phenomena of Fluid Motions", Addison-Wesley Publishing Company, Reading, Mass., 1967, pp. 371-372.

4. L. Rosenhead, "The Steady Two Dimensional Radial Flow of Viacous Fluid Between Two Inclined Plane Walls", Proc. Royal Society A, Vol. 175, p. 436.

5. W. E. Langlois, "Slow Viscous Flow", The Macmillan Company, New York, 1964, pp. 109-110.

6. W. N. Bond, "On Viscous Flow Through Wide-Angled Cones", Philosophical Magazine and Journal of Science, Vol. 50, Sixth Series, 1925, pp. 1058-1066.

7. R. Hill, "The Mathematical Theory of Plasticity", Clarendon Press, Oxford, 1950, pp. 209-212.

8. R. T. Shield, "Plastic Flow in a Converging Conical Channel", Journal of the Mechanics and Physics of Solids, Pergamon Press Ltd., London, Vol. 3, 1955, pp. 246-258.

9. A. G. Fredrickson, "Principles and Applications of Rheology", Prentice-Hall, Inc., Englewood Cliffs, N. J., 1964, Chap. 4.

10. R. Aris, "Vectors, Tensors, and the Basic Equations of Fluid Mechanics", Prentice-Hall, Inc., Englewood Cliffs, N. J., 1962, p. 106.

11. R. B. Bird, "Transport Phenomena", John Wiley and Sons, Inc., New York, N. Y., 1962, pp. 10-15.

12. R. Hill, "New Horizons in the Mechanics of Solids", Journal of the Mechanics and Physics of Solids, 1956, Vol. 5, pp. 66-74.

13. R. J. Tanner, "Non-Newtonian Fluid Parameter Estimation Using Conical Flows", I. and E. C. Fundamentals, Published by American Chemical Society, Washington, D. C., Vol. 5, 1966, pp. 55-69.

14. R. B. Bird, "Transport Phenomena", John Wiley and Sons, Inc., New York, N. Y., 1960, p. 13.

15. M. Reiner, "Deformation and Flow, An Elementary Introduction to Theoretical Rheology", Interscience Publishers, Inc., New York, 1949, pp. 97-98.

16. A. G. Fredrickson, "Principles and Applications of Rheology", Prentice-Hall, Inc., Englewood Cliffs, N. J., 1964, pp. 86-87.

17. G. De Vries, "A Study of Visco and Pseudo-Plastic Flow", Ph. D. thesis-Department of Mechanical Engineering, University of Alberta, Edmonton, Alberta, pp. 37-38.

18. G. De Vries, "A Study of Visco and Pseudo-Plastic Flow", Ph. D. thesis-Department of Mechanical Engineering, University of Alberta, Edmonton, Alberta, pp. 45-48.

19. D. W. Clutter and A. M. O. Smith, "Solution of the General Boundary-Layer Equations for Compressible Laminar Flow, Including Transverse Curveature", Published by Douglas Aircraft Co., Inc., Aircraft Division, Long Beach, Cal., Report No. LB31088, 1963, pp. 34-36.

20. G. De Vries, " A Study of Visco and Pseudo-Plastic Flow", Ph. D. thesis-Department of Mechanical Engineering, University of Alberta, Edmonton, Alberta, p. 95.

21. Sutterby, J. L., "Finite Difference Analysis of Viscous Laminar Converging Flow in Conical Tubes", Applied Scientific Research - Section A, Vol. 15, 1965-66, Martinus Nyhoff, The Hague, pp. 241-252.

22. Bird, R. B., "New Variational Principle for Non-Newtonian Flow", The Physics of Fluids, Vol. 3, 1960, pp. 539-541.

APPENDIX

APPENDIX

AN EXAMPLE COMPUTER PROGRAM

The program developed to obtain the numerical solution to the plane flow problem for a semi-angle of 15 degrees is presented below for reference purposes.

```
      DOUBLE PRECISION PRMT(5), Y(3), DERY(3), AUX (8,3),
      BETA, ALPHA,
     1TN, Y1, YY1, YYY1, YYYY1, BETL, BETH, DEL, V1(31),
      V2(31), V3(31)
     2, P1(31), T1(31), A1, A2, A3, PO, Y2(3,31), YY(3,31),
      YYY(3,31),
     3P(3,31), T(3,31)
      EXTERNAL FCT, OUTP
      COMMON TN,BETA,Y1,PO,Y2,YY,YYY,P,T,LL,LLL
      NDIM=3
C
C     SET PARAMETERS
      TEST=0.2D0
      ND=15
      NL=2*ND+1
      NN=ND*10
      ALPHA=0.15D2
C     CALCULATE BETA FOR A NEWTONIAN FLUID
      BETA=0.4D1/(DCOS(2.*ALPHA*3.14159265358979 3/180.)-0.1D1)
C
C     PREPARE FOR DRKGS
C     SET VALUES OF PRMT AND DERY
      PRMT(1)=0.0D0
      PRMT(2)=ALPHA*3.14159265358979 3/180.
      PRMT(3)=PRMT(2)/FLOAT(NN)
      PRMT(4)=0.1D-5
C     SET VALUE OF EXPONENT
      DO 28 JK=10,100,90
      DO 28 IN=1,10
      TN=FLOAT(11-IN)/FLOAT(JK)
      JC1=1
      JC2=1
      LLL=1
      DEL=0.3D1
      BETL=0.0D0
      BETH=0.0D0
      WRITE(6,100)
      WRITE(6,101) ALPHA,TN
      WRITE(6,108)
   10 CONTINUE
      LL=0
C     SET INITIAL VALUES
      Y(1)=0.1D1
      Y(2)=0.0D0
      Y(3)=BETA
      PO=2**((1.-TN)/2.)*(2./TN*(1.-TN)*Y(1)-1./(2.*TN)*
     1(Y(3)+Y(2)*(TN-1.)/2./(4/*Y(1)**2+Y(2)**2)*(8.*Y(1)
     2*Y(2)+2.*Y(2)*Y(3))))*(4.*Y(1)**2+Y(2)**2)**((TN-1.)/2.)
      DO 11 J=1,3
   11 DERY(J)=0.1D1/0.3D1
C
      CALL DRKGS(PRMT,Y,DERY,DNIM,IHLF,FCT,OUTP,AUX)
C
C     TERMINATE PROCEDURE IF BOUNDARY CONDITION IS SATISFIED
```

```
      IF(DABS(Y1).LT.0.1D-4) GO TO 27
C
C     BEGIN PROCEDURE TO FIND VALUES FOR USE IN INTERPOLA-
      TION FORMULA
      IF(DABS(BETL).GT.0.1D-8.AND.DABS(BETH).GT.0.1D-8)
      GO TO 15
      IF(DABS(Y1).LT.TEST) GO TO 16
   12 IF(Y1) 13,27,14
   13 BETL=BETA
      IF(DABS(BETL).GT.0.1D-8.AND.DABS(BETH).GT.0.1D-8)
      GO TO 15
      BETA=BETA+DEL
      GO TO 10
   14 BETH=BETA
      IF(DABS(BETL).GT.0.1D-8.AND.DABS(BETH).GT.0.1D-8)
      GO TO 15
      BETA=BETA-DEL
      GO TO 10
   15 BETA=(BETL+BETH)/0.2D1
      BETL=0.0D0
      BETH=0.0D0
      DEL=DEL/0.3795D1
      GO TO 10
   16 IF(Y1)17,27,22
   17 IF(JC1+JC2.EQ.5) GO TO 25
      IF(JC1.LT.2) GO TO 18
      IF(JC1.LT.3) GO TO 20
      GO TO 12
   18 YY1=Y1
      IF(LLL.LT3) LLL=LLL+1
      JC1=JC1+1
      IF(JC1+JC2.EQ.5) GO TO 25
      GO TO 12
   20 IF(DABS(Y1-YY1).LT.0.1D-11.OR.DABS(Y1-YYYY1).LT.0.1D-11)
      GO TO 12
      YYY1=Y1
      IF(LLL.LT.3) LLL=LLL+1
      IF(Y1.LT.0.00D0) JC1=JC1+1
      IF(Y1.GT.0.00D0) JC2=JC2+1
      IF(JC1+JC2.EQ.5) GO TO 25
      GO TO 12
   22 IF(JC1+JC2.EQ.5) GO TO 25
      IF(JC2.LT.2) GO TO 23
      IF(JC2.LT.3) GO TO 20
      GO TO 12
   23 YYYY1=Y1
      IF(LLL.LT.3) LLL=LLL+1
      JC2=JC2+1
      IF(JC1+JC2.EQ.5) GO TO 25
      GO TO 12
C
C     CALCULATE COEFFICIENTS FOR INTERPOLATION FORMULA
   25 A1=Y2(2,NL)*Y2(3,NL)/((Y2(1,NL)-Y2(2,NL))*(Y2(1,NL)-
      Y2(3,NL))
```

```
      A2=Y2(1,NL)*Y2(3,NL)/((Y2(2,NL)-Y2(1,NL))*(Y2(2,NL)-
      Y2(3,NL)))
      A3=Y2(1,NL)*Y2(2,NL)/((Y2(3,NL)-Y2(1,NL))*(Y2(3,NL)-
      Y2(2,NL)))
C
C     SUBSTITUTE IN INTERPOLATION FORMULA
      DO 26 J=1,NL
      V1(J)=A1*Y2(1,J)+A2*Y2(2,J)+A3*Y2(3,J)
      V2(J)=A1*YY(1,J)+A2*YY(2,J)+A3*YY(3,J)
      V3(J)=A1*YYY(1,J)+A2*YYY(2,J)+A3*YYY(3,J)
      P1(J)=A1*P(1,J)+A2*P(2,J)+A3*P(3,J)
      T1(J)=A1*T(1,J)+A2*T(2,J)+A3*T(3,J)
      THETA=FLOAT(J-1)/2.
      WRITE(6,102) THETA,VI(J), V2(J), V3(J), P1(J), T1(J)
   26 CONTINUE
      GO TO 28
   27 CONTINUE
      DO 29 J=1,NL
      THETA=FLOAT(J-1)/2.
      WRITE(6,102) THETA, Y2(LLL,J), YY(LLL,J) YYY(LLL,J),
      P(LLL,J),
     1T(LLL,J)
   29 CONTINUE
   28 CONTINUE
  100 FORMAT(1H1,//,55X,'PLANE FLOW',//)
  101 FORMAT(//,40X,'ALPHA=',D8.2,10X,'N=',D10.2,/)
  102 FORMAT(10X,F4.1,6X,5D20.8)
  108 FORMAT(/,10X,'THETA',17X,'M',19X,'MM',17X,'MMM',18X,'P'
     1,19X,'T',/)
      STOP
      END
```

```
 SUBROUTINE FCT(X,Y,DERY)
 DOUBLE PRECISION Y(3), X, DERY(3), C1, C2, TN
 COMMON TN
 DERY(1)=Y(2)
 DERY(2)=Y(3)
 C1=4.0*Y(1)**2+Y(2)**2
 C2=2.*Y(2)*(4.*Y(1)+Y(3))
 DERY(3)=1./(C1+(TN-1.)*Y(2)**2)*((4.*TN**2-12.*TN+4.)
1*Y(2)*C1-((4.*TN**2-6.*TN+2.)*Y(1)+(TN-1.)*Y(3))*C2-
2(TN-1.)*(TN-3.)*Y(2)*C2**2/(4.*C1)-(TN-1.)*Y(2)*(4*
3Y(2)**2+4*Y(1)*Y(3)+Y(3)**2))
 RETURN
 END
```

```
      SUBROUTINE OUTP(X,Y,DERY,IHLF,NDIM,PRMT)
      DOUBLE PRECISION PRMT(5), Y(3), DERY(3), AUX(8,3),
     1BETA, ALPHA, TN, Y1,YY1, YYY1, YYYY1, BETL, BETH,
     2DEL, V1(31), V2(31), V3(31), P1(31), T1(31), A1, A2,
     3A3, PO, Y2(3,31), YY(3,31), YYY(3,31), p(3,31), T(3,31)
      COMMON TN,BETA,Y1,PO,Y2,YY,YYY,P,T,LL,LLL
      IF(DABS(X-LL*PRMT(3)*5).LT.0.1D-3) GO TO  20
      GO TO 24
   20 LL=LL+1
      Y2(LLL,LL)=Y(1)
      YY(LLL,LL)=Y(2)
      YYY(LLL,LL)=Y(3)
      P(LLL,LL)=2**((1.-TN)/2.)*(2./TN*(1.-TN)*Y(1)-1./
     2(2.*TN)*(Y(3)+Y(2)*(TN-1.)/2./(4.*Y(1)**2+Y(2)**2)
     3*(8.*Y(1)*Y(2)+2.*Y(2)*Y(3))))*(4.*Y(1)**2+Y(2)**2)
     4**((TN-1.)/2.)-PO
      T(LLL,LL)=Y(2)*(4.*Y(1)**2+Y(2)**2)**((TN-1.)/2.)
     2*2**((1.-TN)/2.)
   24 IF(DABS(X-PRMT(2)).LT.0.1D-4) GO TO 12
      GO TO 13
   12 Y1=Y(1)
   13 CONTINUE
      RETURN
      END
```